Direct Variation and Proportion

To finish this unit, we will look at different types of variations.

n=1 unless told differently

Definition: A **Direct Variation** is a function in the form $y = Kx^n$

Notes: As x increases, increases y OR as x decreases, decreases y

k is the constant of variation or proportion $\longrightarrow$ $K = \frac{y}{x}$ $\Rightarrow$ $\frac{y_1}{x_1} = \frac{y_2}{x_2}$

Can set up a proportion

Set you know

Set you are missing info.

Example: If y varies directly as x, and $y = 12$ when $x = 20$, find y when $x = 50$.

Method 1

$y = Kx \Rightarrow \frac{12}{20} = \frac{K(20)}{20} \Rightarrow y = 0.6(50)$

$\boxed{y = 30}$

$0.6 = K$

Method 2

$\frac{y_1}{x_1} = \frac{y_2}{x_2} \Longrightarrow \frac{12}{20} = \frac{y_2}{50}$

$\frac{20y_2}{20} = \frac{600}{20}$

Break for Practice: Solve using either method.

1. If y varies directly as x, and $y = 6$ when $x = 4$, find y when $x = 12$.

$y = Kx \longrightarrow 6 = K(4) \longrightarrow y = (3/2)(12)$

$\frac{6}{4} = \frac{4K}{4}$ $\boxed{y = 18}$

$\frac{3}{2} = K$

2. If w varies directly as z, and $w = 4.5$ when $z = 3$, find z when $w = 1.5$.

$W = Kz \longrightarrow 4.5 = K(3) \longrightarrow 1.5 = (1.5)z$

$\frac{4.5}{3} = \frac{3K}{3}$ $\frac{1.5}{1.5} = \frac{1.5z}{1.5}$

$1.5 = K$ $1 = z$

3. If p is directly proportional to q^3, and $p = 3$ when $q = 2$, find p when $q = 4$.

$p = Kq^3 \Rightarrow 3 = K(2^3) \Rightarrow p = \frac{3}{8}(4^3)$

$\frac{3}{8} = \frac{8K}{8}$

$p = \frac{3}{8}(64)$

$\frac{3}{8} = K$

$p = 3(8)$

$\boxed{p = 24}$

4. If x varies directly as $3y + 2$, and $x = 10$ when $y = 6$, find y when $x = 7$.

$x = K(3y+2) \Rightarrow 10 = K(3(6)+2) \Rightarrow 2 \cdot 7 = \frac{1}{2}(3y+2) \cdot 2$

$\frac{10}{20} = \frac{20K}{20}$

$14 = 3y + 2$

$-2 \quad -2$

$\frac{1}{2} = K$

$\frac{12}{3} = \frac{3y}{3}$

$\boxed{4 = y}$

5. If the sales tax on a \$38 purchase is \$2.85, what will the tax be on an \$84 purchase?

Tax = K(cost) $\Rightarrow 2.85 = K(38) \Rightarrow$ Tax = $(0.075)(84)$

$\frac{2.85}{38} = \frac{38K}{38}$

$\boxed{\text{Tax} = \$6.30}$

$0.075 = K$

6. A survey showed that 52 out of 234 people questioned preferred hot cereal to cold. In a school of 1800 people, how many people are likely to prefer hot cereal?

hot = K(total) $\Rightarrow 52 = K(234) \Rightarrow$ hot = $(2/9)(1800)$

$\frac{52}{52} = \frac{234K}{52}$

$\boxed{\text{hot cereal} = 400 \text{ people}}$

$\frac{2}{9}$ or $0.\bar{2} = K$

Extended Practice: Solve using either method

1. If y varies directly as x, and $y = 6$ when $x = 15$, find y when $x = 25$.

$y = Kx \rightarrow \frac{6}{15} = \frac{K(15)}{15} \Rightarrow y = 2/5\,(25)$

$\frac{2}{5} = K$ $\quad \boxed{y = 10}$

OR $K = 0.4$

2. If s is directly proportional to t, and $s = 40$ when $t = 15$, find t when $s = 64$.

$S = Kt \rightarrow \frac{40}{15} = \frac{K(15)}{15} \Rightarrow \frac{3}{8} \cdot 64 = \left(\frac{8}{3}\right) t \cdot \frac{3}{8}$

$\frac{8}{3} = K$ $\quad \boxed{24 = t}$

OR $K = 2.\overline{6}$

3. If p is directly proportional to q, and $p = 9$ when $q = 7.5$, find q when $p = 24$.

$p = Kq \rightarrow \frac{9}{7.5} = \frac{K(7.5)}{7.5} \Rightarrow \frac{24}{1.2} = \frac{1.2q}{1.2}$

$1.2 = K$ $\quad \boxed{20 = q}$

4. If s varies directly as r^2, and $s = 12$ when $r = 2$, find s when $r = 5$.

$S = Kr^2 \Rightarrow 12 = K(2^2) \Rightarrow S = 3(5^2)$

$\frac{12}{4} = \frac{4K}{4}$ $\quad \boxed{S = 75}$

$3 = K$

5. If y is directly proportional to $\sqrt{x}$, and $y = 25$ when $x = 3$, find x when $y = 100$.

$y = K\sqrt{x} \Rightarrow \frac{25}{\sqrt{3}} = \frac{K\sqrt{3}}{\sqrt{3}} \Rightarrow \frac{\sqrt{3}}{\cancel{25}} \cdot \cancel{100}^{4} = \frac{25}{\sqrt{3}}(\sqrt{x}) \cdot \frac{\sqrt{3}}{25}$

$\frac{25}{\sqrt{3}} = K$

$(4\sqrt{3})^2 = (\sqrt{x})^2$

$16 \cdot 3 = x$

$\boxed{48 = x}$

6. If w varies directly as $2x - 1$, and $w = 9$ when $x = 2$, find x when $w = 15$.

$W = K(2x-1) \Rightarrow 9 = K(2(2)-1) \Rightarrow 15 = 3(2x-1)$

$9 = 3K$ $\quad \frac{9}{3} = \frac{3K}{3}$ $\quad 3 = K$

$15 = 6x - 3$ $\quad +3 \quad +3$

$18 = 6x$ $\quad \frac{18}{6} = \frac{6x}{6}$

$\boxed{3 = x}$

7. If the sales tax on a \$60 purchase is \$3.90, what would it be on a \$280 purchase?

$\text{Tax} = K(\text{cost}) \Rightarrow 3.9 = K(60) \Rightarrow \text{Tax} = 0.065(280)$

$\frac{3.9}{60} = \frac{K(60)}{60}$

$0.065 = K$

$\boxed{\text{Tax} = \$18.20}$

8. A real estate agent made a commission of \$5400 on a house that sold at \$120,000. At this rate, what commission will the agent make on a house that sells for \$145,000?

$\text{Commission} = K(\text{price}) \Rightarrow 5400 = K(120{,}000) \Rightarrow \text{Commission} = 0.045(145{,}000)$

$\frac{5400}{120{,}000} = \frac{K(120{,}000)}{120{,}000}$

$0.045 = K$

$\boxed{\text{Commission} = \$6{,}525}$

9. The acceleration of an object varies directly as the force acting on it. If a force of 240 newtons causes an acceleration of 150 m/s^2, what force will cause an acceleration of 100 m/s^2.

$\text{Acceleration} = K(\text{force}) \Rightarrow 150 = K(240) \Rightarrow 100 = 0.625f$

$\frac{150}{240} = \frac{K(240)}{240}$ $\quad \frac{100}{0.625} = \frac{0.625f}{0.625}$

$0.625 = K$

$\boxed{160 \text{ newtons} = \text{force}}$

10. A public-opinion poll found that of a sample of 450 voters, 252 favored a school bond measure. If 20,000 people vote, about how many are likely to vote for the bond measure?

$\text{Favor} = K(\text{total}) \Rightarrow 252 = K(450) \Rightarrow \text{Favor} = 0.56(20{,}000)$

$\frac{252}{450} = \frac{K(450)}{450}$

$0.56 = K$

$\boxed{\text{Favor} = 11{,}200 \text{ people}}$

Inverse and Joint Variations

In the last section we learned what a direct variation is. Now we will learn what an inverse variation is.

Definition: An **Inverse Variation** is a function in the form $y = k/x^n$ ← $n=1$ unless told differently

Note: As x increases, y will decrease OR as x decreases, y will increase
k is the constant of variation or proportion $k = xy \Longrightarrow x_1 \cdot y_1 = x_2 \cdot y_2$

Example: If y varies inversely as x, and $y = 5$ when $x = 4$, find x when $y = 10$.

$y = \frac{k}{x} \Rightarrow 4 \cdot 5 = \frac{k}{4} \cdot 4 \Longrightarrow x \cdot 10 = \frac{20}{x} \cdot x$

$20 = k$

$\frac{10x}{10} = \frac{20}{10}$

$\boxed{x = 2}$

Break for Practice: Solve

1. If a is inversely proportional to b, and $b = 12$ when $a = 8$, find b when $a = 3$.

$a = \frac{k}{b} \Rightarrow 12 \cdot 8 = \frac{k}{12} \cdot 12 \Longrightarrow b \cdot 3 = \frac{96}{b} \cdot b$

$96 = k$

$\frac{3b}{3} = \frac{96}{3}$

$\boxed{b = 32}$

2. If x varies inversely as the square of y, and $x = 2$ when $y = 12$, find y when $x = 8$.

$x = \frac{k}{y^2} \Longrightarrow 2 = \frac{k}{12^2}$

$144 \cdot 2 = \frac{k}{144} \cdot 144$

$288 = k$

$y \cdot 8 = \frac{288}{y^2} \cdot y$

$\frac{8y^2}{8} = \frac{288}{8}$

$\sqrt{y^2} = \sqrt{36} \Longrightarrow \boxed{y = \pm 6}$

It is possible to work with more than one variation at a time. If the term **Jointly** is used, then it means that several variables are varying **directly**.

Break for Practice: Solve.

1. If x varies jointly as y and z, and $x = 100$ when $y = 20$ and $z = 10$, find x when $y = 60$ and $z = 30$.

$x = kyz \rightarrow 100 = k(20)(10) \Longrightarrow x = \frac{1}{2}(60)(30)$

$\frac{100}{200} = \frac{200k}{200}$

$\boxed{x = 900}$

$\frac{1}{2} = k$

2. If x is jointly proportional to y and the square root of z, and $x = 20$ when $y = 5$ and $z = 9$, find x when $y = 6$ and $z = 25$.

$X = Ky\sqrt{z} \Rightarrow 20 = K(5)\sqrt{9} \Rightarrow X = \left(\frac{4}{3}\right)(6)(\sqrt{25})$

$\frac{20}{15} = \frac{15K}{15}$

$\frac{4}{3} = K$

$\boxed{X = 40}$

3. The electrical resistance of a wire varies directly as its length and inversely as the square of its diameter. 100 m of wire with a diameter of 6 mm has a resistance of 12 ohms. Eighty meters of a second wire of the same material has a resistance of 15 ohms. Find the diameter of the second wire.

$R = \frac{K(\text{length})}{\text{diameter}^2} \Rightarrow 6^2 \cdot 12 = \frac{K(100)}{6^2} \cdot 6^2 \Rightarrow d \cdot 15 = \frac{4.32(80)}{d} \cdot d$

$\frac{432}{100} = \frac{100K}{100}$

$4.32 = K$

$\frac{15d^2}{15} = \frac{345.6}{15}$

$\sqrt{d^2} = \sqrt{23.04}$

$\boxed{d = 4.8\text{mm}}$

Extended Practice: Solve

1. If y varies inversely as x, and $y = 3$ when $x = 6$, find x when $y = 18$.

$y = \frac{K}{x} \Rightarrow 6 \cdot 3 = \frac{K}{6} \cdot 6 \Rightarrow x \cdot 18 = \frac{18}{x} \cdot x$

$18 = K$

$\frac{18x}{18} = \frac{18}{18}$

$\boxed{x = 1}$

2. If z is inversely proportional to r, and $z = 32$ when $r = 1.5$, find r when $z = 8$.

$z = \frac{K}{r} \Rightarrow 1.5 \cdot 32 = \frac{K}{1.5} \cdot 1.5 \Rightarrow r \cdot 8 = \frac{48}{r} \cdot r$

$48 = K$

$\frac{8r}{8} = \frac{48}{8}$

$\boxed{r = 6}$

3. If w is inversely proportional to the square of v, and $w = 3$ when $v = 6$, find w when $v = 3$.

$W = \frac{K}{V^2} \Rightarrow 3 = \frac{K}{6^2} \Rightarrow W = \frac{108}{3^2}$

$36 \cdot 3 = \frac{K}{36} \cdot 36$

$W = \frac{108}{9}$

$108 = K$

$\boxed{W = 12}$

4. If p varies inversely as the square root of q, and $p = 12$ when $q = 36$, find p when $q = 16$.

$P = \frac{K}{\sqrt{q}} \Rightarrow 12 = \frac{K}{\sqrt{36}} \Rightarrow P = \frac{72}{\sqrt{16}}$

$6 \cdot 12 = \frac{K}{6} \cdot 6$

$P = \frac{72}{4}$

$72 = K$

$\boxed{P = 18}$

5. If z is jointly proportional to x and y, and $z = 18$ when $x = 0.4$ and $y = 3$, find z when $x = 1.2$ and $y = 2$.

$Z = Kxy \Rightarrow 18 = K(0.4)(3) \Rightarrow Z = 15(1.2)(2)$

$\frac{18}{1.2} = \frac{1.2K}{1.2}$

$\boxed{Z = 36}$

$15 = K$

6. If w is jointly proportional to u and v, and $w = 24$ when $u = 0.8$ and $v = 5$, find u when $w = 18$ and $v = 2$.

$W = Kuv \Rightarrow 24 = K(0.8)(5) \Rightarrow 18 = 6(u)(2)$

$\frac{24}{4} = \frac{4K}{4}$

$\frac{18}{12} = \frac{12u}{12}$

$\boxed{\frac{3}{2} \text{ OR } 1.5 = u}$

$6 = K$

7. If s varies directly as r and inversely as t, and $s = 10$ when $r = 5$ and $t = 3$, for what value of t will $s = 3$ when $r = 4$?

$S = \frac{Kr}{t} \rightarrow 3 \cdot 10 = \frac{K(5)}{3} \cdot 3 \Rightarrow t \cdot 3 = \frac{6(4)}{t} \cdot t$

$\frac{30}{5} = \frac{5K}{5}$ $\quad \frac{3t}{3} = \frac{24}{3}$

$6 = K$ $\quad \boxed{t = 8}$

8. Suppose that r varies directly as p and inversely as q^2, and that $r = 27$ when $p = 3$ and $q = 2$. Find r when $p = 2$ and $q = 3$.

$r = \frac{Kp}{q^2} \Rightarrow 27 = \frac{K(3)}{2^2} \Rightarrow r = \frac{36(2)}{3^2}$

$\left(\frac{4}{3}\right) 27 = \frac{3}{4} K \left(\frac{4}{3}\right)$ $\quad r = \frac{72}{9}$

$36 = K$ $\quad \boxed{r = 8}$

9. The frequency of a radio signal varies inversely as the wave length. A signal of frequency 1200kilohertz (kHz), which might be the frequency of an AM radio station, has wave length 250 m. What frequency has a signal of wave length 400m?

$freq = \frac{K}{wl} \Rightarrow 250 \cdot 1200 = \frac{K}{250} \cdot 250 \Rightarrow freq = \frac{300000}{400}$

$300000 = K$ $\quad \boxed{freq = 750 \text{ KHz}}$

10. The stretch in a wire under a given tension varies directly as the length of the wire and inversely as the square of its diameter. A wire having length 2 m and diameter 1.5 mm stretches 1.2 mm. If a second wire of the same material (and under the same tension) has length 3 m and diameter 2.0 mm, find the amount of stretch.

$Stretch = \frac{K(length)}{d^2} \Rightarrow 1.5^2 \cdot 1.2 = \frac{K(2)}{1.5^2} \cdot 1.5^2$ $\quad Stretch = \frac{1.35(3)}{2^2}$

$\frac{2.7}{2} = \frac{2K}{2}$ $\quad \boxed{Stretch = 1.0125 \text{ mm}}$

$1.35 = K$

Linear Interpolation

We will finish this unit with the topic of linear interpolation. It makes use of ratios and proportions.

Linear Interpolation: A process used to approximate values not given in a table of data. The method is based upon the assumption that within a given interval, the change in one value is proportional to the change in the other value.

Break for Practice: Use linear interpolation and the table to find these values to the nearest tenth.

Year	Bachelor's Degrees awarded in computer science (U.S.)
1968	459
1972	3,402
1976	5,700
1980	11,154
1984	32,172

a. Approximate the number of bachelor's degrees awarded in computer science in 1971.

Year	# of degrees
1968	459
1971	d
1972	3402

3, 4, x, 2943

$\frac{3}{4} = \frac{x}{2943}$

$\frac{8829}{4} = \frac{4x}{4}$

$2207 \approx x$

$d = 459 + 2207$

$\boxed{d = 2666 \text{ degrees}}$

b. Approximate the year that 10,000 bachelor's degrees were awarded in computer science.

Year	# of degrees
1976	5,700
y	10,000
1980	11,154

x, 4, 4300, 5454

$\frac{x}{4} = \frac{4300}{5454}$

$\frac{5454x}{5454} = \frac{17,200}{5454}$

$x \approx 3$ years

$y = 1976 + 3$

$\boxed{y = 1979}$

Example #2: The table gives the temperature in degrees Fahrenheit on a spring day at Chi-Hi

Time (P.M.)	Temperature
1:00	68°
2:00	66°
3:00	63°
4:00	59°
5:00	53°
6:00	45°
7:00	39°

a. Approximate the temperature at 3:40 P.M.

Time	Temp
3:00	63°
3:40	t
4:00	59°

40 mins, 60 mins, x, 4

$\frac{40}{60} = \frac{x}{4}$

$\frac{160}{60} = \frac{60x}{60}$

$x \approx 3°$

temp = 63 − 3

$\boxed{\text{temp} = 60°}$

#'s are always positive

* make sure units are the same

b. At about what time was the temperature 40°?

Time	Temp
6:00	45°
t	40°
7:00	39°

x, 60 mins, 5, 6

$\frac{x}{60} = \frac{5}{6}$

$\frac{300}{6} = \frac{6x}{6}$

50 mins = x

Time = 6:00 + 0:50

$\boxed{\text{Time} = 6:50\text{pm}}$

Extended Practice: Solve each problem using linear interpolation.

1. Consider the table of population figures for the following questions.

Year	U.S. Population in millions
1900	76
1910	92
1920	106
1930	123
1940	132
1950	151
1960	179
1970	203
1980	227
1990	243

a) Approximate the population in 1915.

Year	Population
1910	92
1915	P
1920	106

5, x, 10, 14

$$\frac{5}{10} = \frac{x}{14}$$

$$\frac{70}{10} = \frac{10x}{10}$$

7 million = x

Population = 92 + 7

$\boxed{\text{Population} = 99 \text{ million}}$

b) Approximate the population in 1963.

Year	Population
1960	179
1963	P
1970	203

3, x, 10, 24

$$\frac{3}{10} = \frac{x}{24}$$

$$\frac{72}{10} = \frac{10x}{10}$$

7 million ≈ x

Population = 179 + 7

$\boxed{\text{Population} = 186 \text{ million}}$

c) Approximate the year that the population was 100 million.

Year	Population
1910	92
y	100
1920	106

x, 8, 10, 14

$$\frac{x}{10} = \frac{8}{14}$$

$$\frac{14x}{14} = \frac{80}{14}$$

x ≈ 6 years

Year = 1910 + 6

$\boxed{\text{Year} = 1916}$

d) Approximate the year that the population was 200 million.

Year	Population
1960	179
y	200
1970	203

x, 21, 10, 24

$$\frac{x}{10} = \frac{21}{24}$$

$$\frac{24x}{24} = \frac{210}{24}$$

x ≈ 9 year

Year = 1960 + 9

$\boxed{\text{Year} = 1969}$

2. The table gives the density of dry air at various altitudes.

Altitude (m)	0	500	1000	1500	2000	2500	3000	3500
Density (kg/m³)	1.225	1.167	1.112	1.058	1.007	0.957	0.909	0.863

a) Approximate the density at an altitude of 1200 m.

Altitude | Density

200 [1000 | 1.112] x
1200 | d
500 1500 | 1.058
0.054

$\frac{200}{500} = \frac{x}{0.054}$

$\frac{500x}{500} = \frac{10.8}{500}$

$x \approx 0.022$ Kg/m³

Density = 1.112 − 0.022

$\boxed{\text{Density} = 1.09 \text{ Km/m}^3}$

b) Approximate the density at an altitude of 3200 m.

Altitude | Density

200 [3000 | 0.909] x
3200 | d
500 3500 | 0.863
0.046

$\frac{200}{500} = \frac{x}{0.046}$

$\frac{500x}{500} = \frac{9.2}{500}$

$x \approx 0.018$ Km/m²

Density = 0.909 − 0.018

$\boxed{\text{Density} = 0.891 \text{ Km/m}^3}$

c) Approximate the altitude for dry air with a density of 1.200 kg/m³.

Altitude | Density

x [0 | 1.225] 0.025
a | 1.200
500 500 | 1.167
0.058

$\frac{x}{500} = \frac{0.025}{0.058}$

$\frac{0.058x}{0.058} = \frac{12.5}{0.058}$

$x \approx 216$ m

Altitude = 0 + 216

$\boxed{\text{Altitude} = 216 \text{ m}}$

d) Approximate the altitude for dry air with a density of 0.930 kg/m³.

Altitude | Density

x [2500 | 0.957] 0.027
a | 0.930
500 3000 | 0.909
0.048

$\frac{x}{500} = \frac{0.027}{0.048}$

$0.048x = 13.5$

$x \approx 281$ m

Altitude = 2500 + 281

$\boxed{\text{Altitude} = 2781 \text{ m}}$