Name: ______________________________

Math Investigations I

Unit 7

Standards for Math Investigations I

1. Students will explore historical topics and individuals who have contributed significantly to the development of mathematics.

2. Students will develop an appreciation for math and its relevance in their educational experience.

3. Students will explore non-traditional topics through application of previous mathematical skills and procedures.

4. Students will solve problems by applying and adapting a variety of appropriate strategies and by using appropriate technology.

5. Students will communicate mathematically.

Topic Name	Standard 1	Standard 2	Standard 3	Standard 4	Standard 5
Problem Solving		x		x	x
Mathematicians	x				

This unit is different than the other units in this course. There will be very little in-class lecture. Most of the time you will be given an opportunity to work on a problem set for a given period of time, and afterwards you will have a chance to ask questions. You will be given an opportunity to use the math skills you have learned throughout your mathematics career to try to solve a variety of creative problems. You are encouraged to work with your classmates as you try to figure each problem out.

Good Luck!

Tartaglia (1499-1557) and Cardano (1501-1576)

Probably the most spectacular mathematical achievement of the sixteenth century was the discovery, by Italian mathematicians, of the algebraic solution of cubic and quartic equations. Briefly told, the facts seem to be these:

About 1515, Scipione del Ferro, a professor of mathematics at the University of Bologna, solved algebraically the cubic equation $x^3 + mx = n$, probably being based on earlier Arabic sources. He did not publish his result but revealed the secret to his pupil Antonio Fior. Now about 1535, Nicolo Fontana of Brescia, commonly referred to as Tartaglia (the stammerer) because of a childhood injury that affected his speech, claimed to have discovered an algebraic solution of the cubic equation $x^3 + px^2 = n$. Believing the claim to be a bluff, Fior challenged Tartaglia to a public contest of solving cubic equations, whereupon the latter exerted himself and only a few days before the contest found an algebraic solution for cubics lacking a quadratic term. Entering the contest equipped to solve two types of cubic equations, whereas Fior could solve but one type, Tartaglia triumphed completely.

Later, Girolamo Cardano, an unprincipled genius who taught mathematics and practiced medicine in Milan, upon giving a solemn pledge of secrecy, wheedled the key to the cubic from Tartaglia. In 1545, Cardano published a treatise in which appeared Tartaglia's solution of the cubic. Tartaglia's vehement protests were met by Ludovico Ferrari, Cardano's most capable student, who argued that Cardano had received his information from del Ferro through a third party and accused Tartaglia of plagiarism from the same source. A dispute followed and Tartaglia was lucky to have escaped alive.

Cardano is one of the most extraordinary characters in the history of mathematics. He was born as the illegitimate son of a jurist. He began his life as a doctor, studying, teaching, and writing mathematics while practicing his profession. He was imprisoned for a time for heresy because he published a horoscope of Christ's life. He moved to Rome and became astrologer to the papal court. He died in Rome in 1576, by his own hand, one story says, so as to fulfill his earlier prediction of the date of his death. Many stories are told of his wickedness, as when in a fit of rage, he cut off the ears of his younger son.

Tartaglia had a hard childhood. When the French took his home town of Brescia, Tartaglia and his father fled with many others into the cathedral for sanctuary, but the soldiers pursued and a massacre took place. The father was killed and the boy, with a split skull and a severe saber cut that cleft his jaws and palate, was left for dead. Later his mother found the boy and carried him off. Lacking resources for medical assistance, she recalled that a wounded dog always licks the injured spot, and Tartaglia later attributed his recovery to this remedy. The injury to his palate caused a lifetime imperfection in his speech, from which he received his nickname of "the stammerer."

His mother gathered together sufficient money to send him to school for fifteen days, and he made the best of the opportunity by stealing a copybook from which he subsequently taught himself how to read and write. It is said that lacking the means to buy paper, he was obliged to use tombstones in the cemetery as slates.

Problem Set 1: Intuition vs Proof – Answer the following intuitively, and then check your answers by calculation.

1.) A car travels from P to Q at the rate of 40 mph and then returns from Q to P at the rate of 60 mph. What is the average rate for the round trip?

2.) A can do a job in 4 days, and B can do the same job in 6 days. How long will it take A and B together to do the job?

3.) A man sells half of his apples at 3 apples for 17 cents, and then sells the other half at 5 apples for 17 cents. At what rate should he sell all of his apples to make the same income?

4.) A clock strikes six in 5 seconds. How long will it take to strike twelve?

5.) A bottle and cork cost $1.10. If the bottle costs a dollar more than the cork, how much does the cork cost?

6.) Suppose that in 1 glass there is a certain quantity of a liquid A, and in a second glass there is an equal quantity of another liquid B. A spoonful of liquid A is taken from the first glass and put into the second glass, then a spoonful of the mixture from the second glass is put back into the first glass Is there now more or less liquid A in the second glass than there is liquid B in the first glass?

7.) A boy wants the arithmetical average of his 8 grades. He averages the first 3 grades, then the last 5 grades, and then finds the average of these averages. Is this correct?

Napier

Many of the fields in which numerical calculations are important, such as astronomy, navigation, trade, engineering, and war, made ever-increasing demands that these computations be performed more quickly and accurately. These increasing demands were met successively by four remarkable inventions: the Hindu-Arabic notation, decimal fractions, logarithms, and computing machines.

John Napier invented logarithms in the early 17th century. Born when his father was only 16 years of age, John Napier lived most of his life at the imposing family estate of Merchiston Castle, near Edinburgh, Scotland, and expended much of his energies in the political and religious controversies of his day. He was violently anti-Catholic, and published a bitter and widely read attack on the Church of Rome in which he endeavored to prove that the Pope was the Antichrist, and that the Creator proposed to end the world in the years between 1688 and 1700.

Napier also wrote prophetically of various infernal war engines, accompanying his writings with plans and diagrams. He predicted the future would develop a piece of artillery that could “clear a field of four miles’ circumference of all living creatures exceeding a foot of height”, “that it would produce “devices for sayling under water,” and that it would create a chariot with “a living mouth of mettle” that would “scatter destruction on all sides.” In World War I, these were realized as the machine gun, the submarine, and the army tank, respectively.

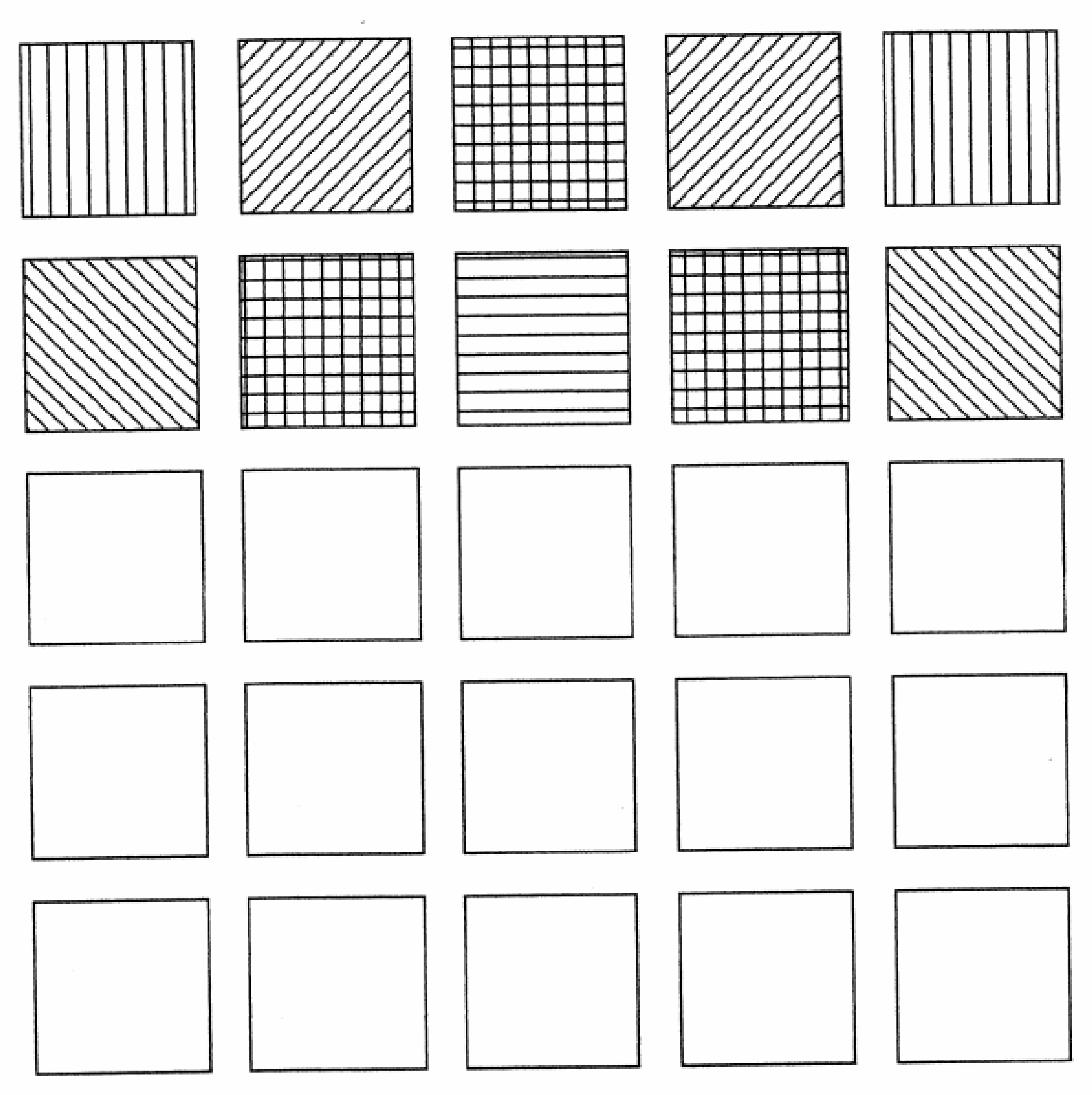
Many stories are told of Napier. Once he announced that his coal-black rooster would identify for him which of his servants was stealing from him. The servants were sent one by one into a darkened room with instructions to pat the rooster on the back. Unknown to the servants, Napier had coated the bird’s back with lampblack, and the guilty servant, fearing to touch the rooster, returned with clean hands. There was also the occasion when Napier became annoyed with his neighbor’s pigeons eating his grain. He threatened to impound the birds if his neighbor did not restrict their flight. The neighbor, believing the capture of his pigeons to be virtually impossible, told Napier that he was welcome to the birds if he could catch them. The next day, the surprised neighbor observed the pigeons staggering on Napier’s lawn with Napier calmly collecting them into a large sack. Napier had rendered the birds drunk by scattering some brandy-soaked peas about his lawn.

As a diversion from his politics and religion, Napier amused himself with the study of mathematics. The four products of his genius are

1) the invention of logarithms
2) a clever mnemonic for reproducing formulas for solving right spherical triangles
3) at least two trigonometric formulas useful in the solution of oblique spherical triangles
4) the invention of Napier rods used for mechanically multiplying, dividing, and taking square roots of numbers.

Problem Set 2: *This activity on patterns will be done as a class activity.*

Two of five rows of geometrical figures are completed below. Complete row 3 and compare with the figures shown to the class by your teacher. Then complete rows 4 and 5 when directed by the teacher.



Pascal

Blaise Pascal was born in France in 1623 and very early showed phenomenal ability in mathematics. Because of his delicate physical condition, he was kept at home to ensure his not being overworked. His father restricted his studies to languages. However, the exclusion of mathematics so aroused the boy's curiosity that he inquired of his tutor the nature of geometry. Stimulated by the tutor's description of the subject, he gave up his playtime and secretively discovered for himself many properties of geometric figures. When his father came upon his boy during his geometric activities, he was so struck by his son's abilities that he gave his son a copy of Euclid's *Elements*, which the boy quickly mastered.

At the age of 14 he participated in the weekly gatherings of a group of French mathematicians. When he was 18 or 19 he invented the first calculating machine to assist his father in the auditing of government accounts. He was to manufacture over fifty calculating machines, some of which are still preserved in Paris. At 21 he applied his talents to physics and discovered the principle of hydrostatics known today as Pascal's principle.

At 27 this mathematical productivity ceased when, suddenly frail, Pascal devoted himself entirely to religious contemplation. Three years later he returned briefly to mathematics, and in association with Fermat, laid the foundation for the mathematical theory of probability. But one year later he received what he regarded as a strong intimation that these renewed activities were not pleasing to God. The divine hint occurred when his runaway horses plummeted over a bridge and he narrowly survived.

Only once again did he return to mathematics. While suffering a toothache, some geometrical ideas occurred to him and his toothache suddenly ceased. Regarding this a sign of divine will, Pascal plunged into mathematics for eight days producing some remarkable mathematics including some problems that subsequently, when issued as challenge problems, baffled other mathematicians. He died at the age of 39 and is described as the "greatest might have been" in the story of mathematics.

Problem Set 3:

1.) Use Guess and Check on the following problems. **Explain your reasoning.**

a.) What is the length of this room?

b.) How high is the ceiling in this room?

c.) How many times does your heart beat in one hour?

d.) How many breaths do you take in one day?

e.) A tour group consisting of six married couples is visiting the Washington Monument. The elevator to the top has a load limit of 1000 kg. Is it safe for all six couples to ride the elevator at once?

f.) How long would it take to count to one million if you counted for eight hours a day, five days a week?

Leibniz (1646-1716)

Gottfried Wilhelm Leibniz was born in Leipzig, Germany in 1646. Having taught himself to read Latin and Greek when he was a mere child, he had, before he was 20, mastered the ordinary textbook knowledge of mathematics, philosophy, theology, and law. He was a particularly gifted linguist, winning some fame as a Sanskrit scholar, but he spent most of his life in diplomatic service. He tried without success to reunite the Protestant and Catholic churches. He also felt he might have a way of Christianizing all of China by what he believed to be the image of creation in the binary arithmetic. Since God may be represented by unity and nothing by zero, he imagined that God created everything from nothing. He had hoped to convert the reigning Chinese emperor (who was particularly attached to science) and thence all of China. He also remarked that imaginary numbers are like the Holy Ghost, midway between existence and nonexistence.

Leibniz is credited with the invention of calculus along with Sir Isaac Newton, although it is his symbolism and not Newton's that we use today. The theory of determinants is said to have originated with Leibniz. If Leibniz was not as penetrating a mathematician as Newton, he was perhaps a broader one, and although inferior to his English rival as an analyst and mathematical physicist, he probably had a keener mathematical imagination and a superior instinct for mathematical form.

There are two broad and opposite domains of mathematical thought, the continuous and the discrete. Leibniz is the one man in the history of mathematics who possessed both of these qualities of thought to a superlative degree.

Problem Set 4:

1.) "I am thinking of two whole numbers," said Mariam. "When I add them, the answer is 45, but when I multiply them the answer is 486. What are my numbers?"

2.) The sum of two whole numbers is 47, and their difference is a whole number less than 10. Find the solution set.

3.) At a supermarket, apples cost $0.33, and pears cost $0.55 each. If you buy 23 pieces of fruit for a basket of apples and pears as a present for a friend, you pay $9.57. How many of each did you buy?

4.) Three consecutive odd integers have a sum of 399. What are the numbers?

Isaac Newton (1642-1727)

Isaac Newton was born in Woolsthorpe hamlet in England on Christmas Day, 1642, the year in which Galileo died. His father, who was a farmer, died before he was born, and it was first planned that his son would also be a farmer. The youngster, however, showed great delight in making clever devices, such as a toy gristmill that ground wheat to flour, with a mouse serving as motive power, and a wooden clock that worked by water. At Trinity College his attention was directed toward mathematics. He read Euclid's *Elements*, that he found too obvious, and Descartes' geometry book, which he found somewhat difficult. He discovered the general binomial theorem, created calculus, performed experiments in optics, and formulated the basic principles of his theory of gravity.

He lectured at Cambridge University for 18 years and was vehemently attacked by some scientists for a paper he published on his theory of colors. Consequently, Newton did not publish most of his findings until years after their discovery. Because of this postponement arose the bitter dispute with Leibniz over the discovery of calculus. It was owing to this controversy that English mathematicians, backing Newton, cut themselves off from continental developments, and mathematical progress in England was retarded for practically 100 years.

With Halley (Halley's comet) paying the cost, Newton published his book on the principles governing the universe, which made a tremendous impact throughout Europe. It proved to be the most influential and most admired work in the history of science. In 1692, he suffered a curious illness that lasted two years and involved some sort of mental derangement. Most of his later life was devoted to chemistry, alchemy, and theology.

Newton was never beaten by any of the various challenge problems that circulated among the mathematicians of his time. As a mathematician he is ranked almost universally as the greatest the world has yet produced. His insight into physical problems and his ability to treat them mathematically have probably never been excelled.

His greatness can further be illustrated by many testimonials, such as by Leibniz, who said, "Taking mathematics from the beginning of the world to the time when Newton lived, what he did was much the better half." And there is the remark made by Lagrange to the effect that Newton was the greatest genius that ever lived, and the most fortunate, for we can find only once a system of the universe to be established.

Newton's own estimate of his work: "I do not know what I may appear to the world, but to myself I seem to have been only like a little boy playing on the seashore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me." In generosity to his predecessors, he explained that if he had seen farther than other men, it was only because he stood on the shoulders of giants.

Problem Set 5:

1.) Find the sum of the numbers 1 to 100.

2.) A piece of paper is .003 inches thick. If it is folded to have a thickness of 2 pieces, then folded again to a thickness of 4 pieces and so on again and again, how thick would the stack of paper be after 50 folds?

3. Consider a 10-inch by 10-inch square piece of paper. It is folded in half to form a rectangle and then folded again to form a square. Then this process is repeated over and over again. What would be the dimensions of the paper after 50 folds?

Challenge:
How many segments can be drawn between 3 points if each pair of points must be connected? How many segments if 4 points are used? 5 points? 10 points? 100 points?

Descartes (1596-1650) and Fermat (1601-1665)

Rene Descartes was born near Tours, France in 1596. At the age of eight he was sent to a Jesuit school where he developed his lifelong habit (at first because of delicate health) of lying in bed until late in the morning. He went to Paris when he was 16 and devoted some time to the study of mathematics. Five years later he joined the army. Upon quitting military life, he traveled considerably in Europe before settling in Holland at the height of his power. There he lived for twenty years, devoting his time to philosophy, mathematics, and science. In 1649 he reluctantly went to Sweden at the invitation of Queen Christina. A few months later, he contracted inflammation of the lungs and died in Stockholm. The great philosopher-mathematician was entombed in Sweden and efforts to have his remains shipped to France failed. Then, 17 years after Decartes' death, his bones, except for those of his right hand, were returned to France and reinterred in Paris. The bones of the right hand were secured, as a souvenir, by the French Treasurer-General who had arranged the transportation of the bones.

While in Holland for 20 years, Descartes accomplished his writing. He spent four years writing a physical account of the universe, but abandoned it when he heard of Galileo's condemnation by the Church. He advanced algebraic geometry over the Greeks who represented a variable as a length of some line segment. As a consequence, he developed the Cartesian Coordinate System that we use today which is the basis of analytic geometry.

Pierre de Fermat was another French mathematical genius who is credited along with Descartes for formulating the basis of analytic geometry. Meanwhile, his contributions toward probability were done jointly with Pascal. Fermat was the son of a leather merchant and received his early education at home. Working as a humble and retiring lawyer, he devoted the bulk of his leisure time to the study of mathematics. Although he published very little in his lifetime, he was in correspondence with many leading mathematicians and thus influenced considerably his contemporaries. He is considered the greatest French mathematician of the 17th century.

His most outstanding contribution is the founding of the modern theory of numbers. Many of his contributions in this field occurred as marginal statements in a book. Here are some examples:

1) If p is prime and a is prime to p, then a^{p-1} is divisible by p.
   This theorem was given by Fermat without proof and wasn't proven for 100 years.
2) A prime of the form 4n + 1 can be represented as the sum of two squares.
   The first published proof came over 100 years later.
3) There do not exist positive integers x, y, z, n such that $x^n + y^n = z^n$, when n>2.
   This is known as Fermat's Last Theorem. Fermat claimed that he had a remarkable proof, but the margin is too narrow to contain it. This theorem has the distinction for which the greatest number of incorrect solutions have been published.

Of all the problems that Fermat said he had proofs for, all have since been proven although there is some controversy over the proof of Fermat's Last Theorem. It is also interesting to note that his conjecture that $2^{2^n} + 1$ is prime was shown incorrect.

Fermat also contributed ideas which eventually led to the discovery of Calculus.

Problem Set 6:

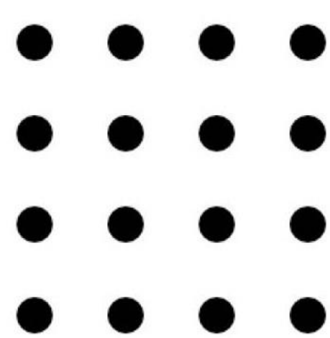
1.) The surface of a 3-inch cube is painted and the cube is then cut into 1-inch cubes. How many of the 1-inch cubes have paint on only 1 face? On 2 faces? 3 faces? 4 faces?

2.) Using 4 cuts, what is the greatest number of pieces of pizza you can get? The cuts have to be straight from edge to edge. How many pieces in 10 cuts?

3.) A rope is just long enough to wrap around the earth's equator exactly once. How much rope would have to be added so that it is raised 1 foot above the earth's surface all the way around?

4.) Divide the number 45 into four parts such that when 2 is added to the first part, 2 is subtracted from the second part, 2 is multiplied by the third part, and the fourth part is divided by 2; the four results will be the same number.

5.) Draw six line segments through the 16 points without lifting your pencil off the paper or retracing a line.



Problem Set 7:

1.) Complete each pattern without use of your calculator.

a.) 10, 6, 2, ________, ________, ________

b.) 1, 2, 6, 24, ________, ________, ________

c.) 7.5, 6.6, 5.7, 4.8, ________, ________, ________

d.) 1, 5, 6, 11, 17, ________, ________, ________

e.) $\sqrt{1 + 1 \cdot 2 \cdot 3 \cdot 4} = 5$

$\sqrt{1 + 2 \cdot 3 \cdot 4 \cdot 5} = 11$

$\sqrt{1 + 3 \cdot 4 \cdot 5 \cdot 6} = 19$

$\sqrt{1 + 4 \cdot 5 \cdot 6 \cdot 7} = 29$

$\sqrt{1 + 50 \cdot 51 \cdot 52 \cdot 53} =$ ___________

2.) Find the sum of $1 + 3 + 5 + \cdots + (2n - 1)$. [Note: This is not an infinite number of terms.]

3.) Twelve people attend a dinner. Each person shakes hands with all the other people. What is the total number of handshakes?

4.) Find the number of angles in each figure. How many angles would the 20^{th} figure contain?

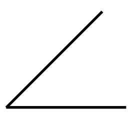


Fig. 1

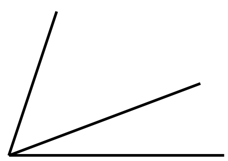


Fig. 2

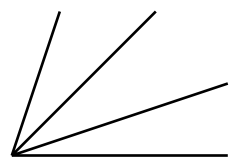


Fig. 3

Challenge:

Find the sum of $\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+\cdots+\frac{1}{99\times100}$

Problem Set 8:

1.) Complete each pattern.

a.) (16, 4), (9, 3), (4, 2), (______, ______), (______, ______), (______, ______)

b.) 1, 3, 6, 10, ________, ________, ________

c.) 1, 8, 27, 64, ________, ________, ________

2.) A number has 6 digits. The first 5 digits are 30644. What is the one's digit if the number is divisible by 3?

3.) A company offers its employees a choice of two different salary packages. The employees may choose a 5% increase in salary the first year followed by a 10% increase in salary the second year, or they may choose a 10% increase in salary the first year followed by a 5% increase the second year. If you were an employee at this company, which salary package would you choose and why?

4.) A teacher drives from Al Ain to Abu Dhabi at 80 kilometers per hour but makes an immediate return from Abu Dhabi to Al Ain at 120 kilometers per hour. What was his/her average speed for the entire trip?

5.) The following figures represent rectangular numbers. How many points would figure 10 contain? Figure 100?

Fig. 1

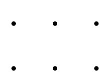


Fig. 2

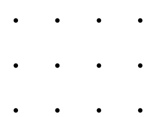


Fig. 3

Challenge:

Mary is 24 years old. Mary is twice as old as Anne was when Mary was as old as Anne is now. How old is Anne?

Problem Set 9:

1.) Arrange the numbers 1, 2, 3, and 4 in 4 rows and 4 columns so that no number appears in each row and column more than once.

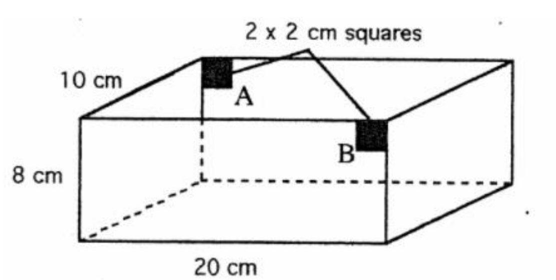
2.) A frog fell into a well that was 10 feet deep. Each day he would climb up 3 feet but during the night would fall back 2 feet. At this rate, when would the frog get out of the well?

3.) Mona had a dinner for some friends. Everyone shook hands with everyone else. How many people were at the party if there were 78 handshakes?

4.) How many squares (of different sizes) are there on a regular 8 by 8 chessboard?

5.) In the expansion of xy^2, the values of x and y are each increased by 25%. What is the new value of xy^2 in comparison to its old value?

Challenge:
An ant is situated on the inside of an open box at point A as shown in the diagram. Find the shortest distance that the ant would need to walk to get to point B which is inside but on the opposite side of the box.



Problem Set 10:

1.) How many cubes (of different sizes) are there on an 8 by 8 by 8 Rubik's Cube?

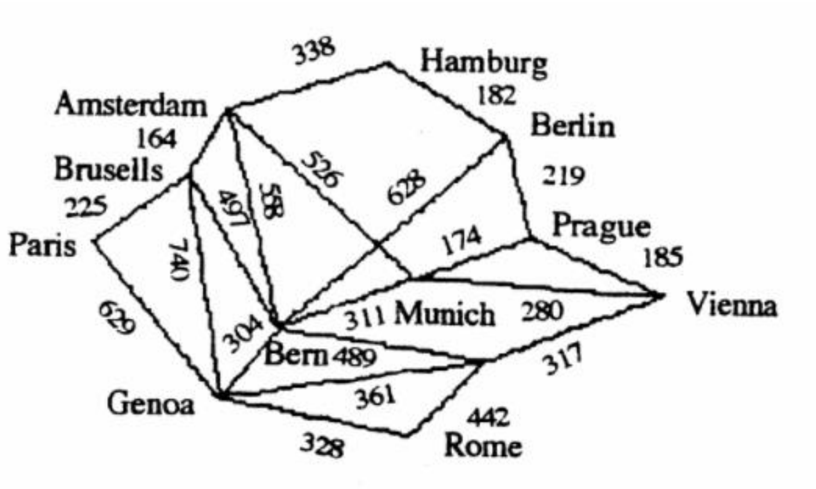
2.) Radioactive carbon, ^{14}C , is often used to date fossils. ^{14}C decays with a half-life of 5600 years. This means that after 5600 years, half of the quantity decays and half remains. If a fossil insect is found in rock that has only $\frac{1}{16}$ the usual amount of ^{14}C, how old is the fossil?

3.) Alice, Bob, Chuck, David and Elsa live in five different houses that are numbered 101, 102, 103, 104, and 105, along a street that runs from south to north, with 105 being farthest north. Furthermore

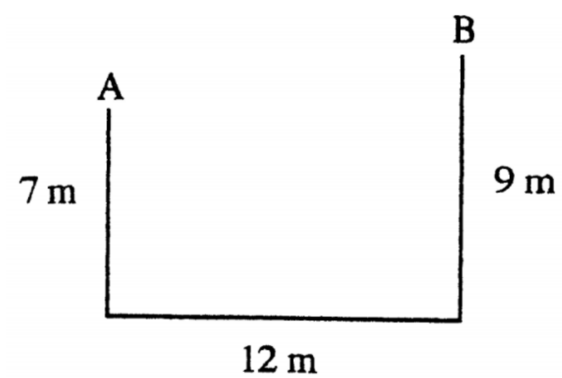
Alice does not live in 105
Bob does not live in 101
Chuck does not live in either 101 or 105
David lives in a house farther north than Bob
Elsa does not live in a house adjacent to Chuck
Chuck does not live in a house adjacent to Bob

Identify the occupants of the five houses 101 through 105.

4.) Find the path with the shortest distance from Paris to Vienna.



Challenge:
A young deer at point A is 7 m from a river bank. What is the shortest distance that it needs to walk to reach the river bank for a drink of water and then proceed to point B?



Problem Set 11:

1.) One hundred lockers in a row are all initially closed. One student opens every locker followed by a second student who closes every other locker starting with the second locker. A third student then changes every third locker by either closing or opening it. A fourth student similarly changes every fourth locker, a fifth does likewise for every fifth locker, etc. This process is continued until the 100^{th} student changes the last locker. At the conclusion of this process, how many lockers remain open?

2.) Three men are seated around a circular table facing each other. They are told that a box in the room contains five hats – three white and two black. A hat is placed on each man's head. The remaining two hats are unseen, and no man sees that hat that is placed on his own head.

One man is then asked after a while what color hat he believes to be on his head. He looks at the other two hats, and then says he doesn't know. The second man likewise didn't know for sure what color hat he had on. The question is then put once again, and the third man, who is **BLIND**, correctly announces the color of the hat that is on his head. What color was the hat and how did he know what color it was?

3.) The three peg puzzle first appeared as a toy in 1883 in France. Shortly after it was introduced, the following story, as told by W.W.R. Ball in *Mathematical Recreations and Essays*, was associated with it. As a result, the game became known as the Tower of Brahma. It is also widely known as the Tower of Hanoi.

In the great temple at Benares, beneath the dome which marks the center of the world, rests a brass plate in which are fixed three diamond needles, each a cubit high and as thick as the body of a bee. On one of these needles, at the creation, God placed sixty-four disks of pure gold, the largest disk resting on the brass plate, and the others getting smaller and smaller up to the top one. This is the Tower of Brahma. Day and night unceasingly the priests transfer the disks from one diamond needle to another according to the fixed and immutable laws of Brahma, which require that the priest on duty must not move more than one disk at a time and that he must place this disk on a needle so that there is no smaller disk beneath it. When the sixty-four disks shall have been thus transferred from the needle on which at the creation God placed them to one of the other needles, tower, temple, and Brahmins alike will crumble into dust, and with a thunderclap the world will vanish.

What is the smallest number of moves by which the priests can successfully transfer all 64 rings from one needle to another according to the given rules?