Algebra II
Unit 5: Rational Expressions

Priority Standard: A.APR.7d – Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication and division by a nonzero rational expression: add, subtract, multiply and divide rational expressions.

Unit "I can" statements:

1. I can simplify quotients using the laws of exponents.
2. I can simplify expressions involving exponents of zero and negative integers.
3. I can correctly use scientific notation and significant digits.
4. I can simplify rational algebraic expressions.
5. I can graph rational functions.
6. I can multiply and divide rational expressions.
7. I can add and subtract rational expressions.
8. I can solve equations and inequalities that have fractional coefficients.
9. I can solve fractional equations.

Common Core State Standards that are addressed in this unit include: A.SSE.1a, A.SSE.1b, A.CED.1a, A.REI.2a, A.APR.3, A.APR.7d
For more information see www.corestandards.org/Math/

Quotients of Monomials

In this unit we will explore rational algebraic expressions. Since rational numbers are those that can be written as fractions, then rational algebraic expressions must involve both fractions and algebraic expressions. In this unit we will extend many of the same skills that you learned with plain fractions to rational algebraic expressions. We will begin with simply reducing fractions.

Review: Reduce each fraction.

1. $\frac{16}{56}$ $(\div 8) = \frac{2}{7}$

2. $\frac{75}{125}$

3. $\frac{63}{81}$

In order to reduce rational algebraic expressions, we need two more laws of exponents.

Example	Law of Exponents
$\left(\frac{2}{5}\right)^3 = \left(\frac{2}{5}\right)\left(\frac{2}{5}\right)\left(\frac{2}{5}\right) = \frac{2^3}{5^3} = \frac{8}{125}$	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$
$\frac{x^5}{x^3} = \frac{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot x \cdot x}{\cancel{x} \cdot \cancel{x} \cdot \cancel{x}} = x^2$	$\frac{a^m}{a^n} = a^{m-n}$ If $m > n$
$\frac{x^3}{x^5} = \frac{\cancel{x} \cdot \cancel{x} \cdot \cancel{x}}{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot x \cdot x} = \frac{1}{x^2}$	$\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$ If $n > m$

Now we will use these laws to reduce fractions involving algebraic expressions.

Note: A quotient is reduced when:
1. All common factors are taken out.
2. Each base appears only once.
3. All parentheses are gone.

Break for Practice: Simplify.

1. $\frac{18y}{4y^3} = \frac{9}{2y^2}$

2. $\frac{15p^3q^2}{18p^2q^5} = \frac{5p}{6q^3}$

3. $\frac{-8s^4t}{16st^4} = \frac{-1s^3}{2t^3}$

4. $\frac{6x^2}{y^2} \cdot \frac{x}{y^2} = \frac{6x^3}{y^4}$

$\cdot \frac{8x^2}{15y^3} = \frac{40x^3y}{30y^3}$

$= \frac{4x^3}{3y^2}$

6. $\frac{p^3}{q}\left(\frac{3q}{p}\right)^2 = \frac{p^3}{q} \cdot \frac{9q^2}{p^2}$

$= \frac{9p^3q^2}{p^2q}$

$= 9pq$

7. $\frac{(x^2y^3)^2}{(x^3y)^2} = \frac{x^4y^6}{x^6y^2}$

$= \frac{y^4}{x^2}$

8. $\frac{(ab^2c)^2}{(a^3bc^2)^3} = \frac{a^2b^4c^2}{a^9b^3c^6}$

$= \frac{b}{a^7c^4}$

Extended Practice: Simplify

1. $\frac{-12p^3q}{4p^3q^2} = \frac{-3}{q}$	2. $\frac{30x^2y^3}{-6x^3y^2} = \frac{-5y}{x}$
3. $\left(\frac{3r}{s^2}\right)^3 = \frac{27r^3}{s^6}$	4. $\left(\frac{2x^2}{-y}\right)^4 = \frac{16x^8}{y^4}$
5. $\frac{3x^2}{y^2} \cdot \frac{3y}{6x} = \frac{3x}{2y}$	6. $\frac{xy^2}{2} \cdot \frac{6x}{y^2} = 3x^2$
7. $\frac{(xyz^2)^2}{(x^2yz)^2} = \frac{z^2}{x^2}$	8. $\frac{(pq^2r^3)^3}{(p^3qr^2)^2} = \frac{q^4r^5}{p^3}$

9. $\frac{u^2}{v}\left(\frac{3v}{u^2}\right)^2 = \frac{9v}{u^2}$	10. $\left(\frac{2x^2}{y^3}\right)^3\left(\frac{-y^3}{2x^2}\right)^2 = \frac{2x^2}{y^3}$

Zero and Negative Exponents

In this section we will see how two new properties of exponents can help us simplify rational algebraic expressions.

Property	Explanation
$a^0 = 1$	Consider: $\frac{x^2}{x^2} = 1$ And: $\frac{x^2}{x^2} = x^{2-2} = x^0$ $\therefore\ x^0 = 1$
$a^{-n} = \frac{1}{a^n}$ Note: $\frac{1}{a^{-n}} = a^n$	Consider: $\frac{x^3}{x^5} = \frac{x \cdot x \cdot x}{x \cdot x \cdot x \cdot x \cdot x} = \frac{1}{x^2}$ And: $\frac{x^3}{x^5} = x^{3-5} = x^{-2}$ $\therefore\ x^{-2} = \frac{1}{x^2}$

Break for Practice: Write in simplest form without negative or zero exponents. (★ Re-arrange so that there are no negative exponents then simplify)

1. $4 \cdot 3^{-2} \Rightarrow \frac{4}{3^2} = \frac{4}{9}$

2. $(4^{-1} \cdot 6^{-1} \cdot 7^0)^{-1}$
$4^1 \cdot 6^1 \cdot 7^0 \Rightarrow 4 \cdot 6 = 24$

3. $\left(\frac{5}{2}\right)^{-2} \Rightarrow \frac{5^{-2}}{2^{-2}} = \frac{2^2}{5^2} = \frac{4}{25}$

4. $\frac{s^{-3}t^{-4}}{s^{-2}t^0} \Rightarrow \frac{s^2}{s^3t^4} = \frac{1}{st^4}$ ($t^0 = 1$)

5. $\left(\frac{2}{x^2y^{-3}}\right)^{-2} \Rightarrow \frac{2^{-2}}{x^{-4}y^6} = \frac{x^4}{2^2y^6} = \frac{x^4}{4y^6}$

6. $\frac{9ab^{-2}}{-3a^{-3}b^{-1}} \Rightarrow \frac{9a \cdot a^3 \cdot b}{-3 \cdot b^2} = \frac{-3a^4}{b}$

7. $\frac{(2a^{-2}b)^{-3}}{ab^4}\left(\frac{a^{-2}}{b^{-3}}\right)^{-1} \Rightarrow \frac{2^{-3}a^6b^{-3}}{ab^4} \cdot \frac{a^2}{b^3} = \frac{a^6}{2^3 \cdot b^3ab^4} \cdot \frac{a^2}{b^3}$
$= \frac{a^8}{8ab^{10}}$
$= \frac{a^7}{8b^{10}}$

expressions are considered simplified if no fractions are used.

ior Practice: Write without using fractions. No $\div$ — / (Division) Multiplication ONLY

1. $\frac{a^3}{b^2} \rightarrow a^3 \cdot b^{-2}$

2. $\frac{6x^3}{yz^2} \rightarrow 6x^3y^{-1}z^{-2}$

Extended Practice: Write in simplest form without zero or negative exponents.

1. $3 \cdot 5^{-1} = \frac{3}{5}$	2. $(3 \cdot 5)^{-1} = \frac{1}{15}$
3. $(-3^{-1})^{-2} = 9$	4. $2\left(\frac{2}{5}\right)^{-2} = \frac{25}{2}$
5. $\frac{p^{-1}q^{-2}}{p^{-3}} = \frac{p^2}{q^2}$	6. $\frac{s^{-2}t^{-3}}{s^{-1}t^0} = \frac{1}{st^3}$
7. $\frac{6xy^{-1}}{-2x^{-2}y^{-1}} = -3x^3$	8. $\left(\frac{u^{-2}}{v}\right)^{-1} = u^2v$
9. $\left(\frac{2}{h^2k^{-3}}\right)^{-2} = \frac{h^4}{4k^6}$	10. $\frac{(3x^{-2}y)^{-1}}{(2xy^{-2})^0} = \frac{x^2}{3y}$
11. $\left(\frac{a^0}{b}\right)^{-2}\left(\frac{a}{b^{-2}}\right)^{-2}(ab^2)^{-1} = \frac{1}{a^3b^4}$	

12. $\left(\frac{17x^{-2}y^{-3}z^{4}}{34x^{-8}y^{7}z^{-3}}\right)^{0} = 1$

Write without using fractions.

1. $\frac{6x^3}{y^3} = 6x^3y^{-3}$	2. $\frac{x^3}{yz^4} = x^3y^{-1}z^{-4}$

Scientific Notation and Significant Digits

One common place where negative exponents are found is in scientific notation. What do you already know about scientific notation?

Definition:

A number is in **scientific notation** when it is in the form $m \times 10^n$ $(1 \le m \le 10)$

"m" is called the mantissa

"n" is called the Characteristic

Break for Practice: Complete the chart.

Rewrite in decimal form	Rewrite in Scientific Notation
$3.17 \times 10^5 = 317{,}000$	$32{,}000 = 3.2 \times 10^4$
$5.143 \times 10^{-2} = 0.05143$	$0.0001084 = 1.084 \times 10^{-4}$
$-2.5 \times 10^3 = -2{,}500$	$39648.1 = 3.96481 \times 10^4$
$-3.4 \times 10^{-1} = -0.34$	$0.1048 = 1.048 \times 10^{-1}$
$10^3 = 1000$	$500 = 5 \times 10^2$
$10^{-4} = 0.0001$	$4{,}151{,}964{,}000{,}000 = 4.151964 \times 10^{12}$

In many applications, it is important to be aware of significant digits. These are very important in the sciences. When a number is in scientific notation, the number of digits in the mantissa is the number of significant digits. When a number is in decimal form, it is a little more complicated.

Rules for counting significant digits:

1. All non-zero digits are significant.

   Example: 2.3456

2. All zeros between non-zero digits are significant.

   Example: 3.408 , 340.8 , 340.8

3. All zeros which are at the same time to the right of the decimal point and at the end of the number are significant.

   significant

   Example: 0.0034500

   place holding (not significant)

4. All zeros which are to the left of a written decimal point and are in a number greater than or equal to 10 are significant.

   Example: 5,000 no decimal 5,000.

Break for Practice: Identify the number of significant digits.

Number	Number of Significant Digits
48,923	5
3.967	4
900.06	5
0.0004	1
8.1000	5
501.040	6
3,000,000	1
3,000,000.	7
10.0	3

Extended Practice: Complete the chart. Count the number of significant digits in the original form of the number.

Decimal Notation	Scientific Notation	Number of Significant Digits
7,500	7.5×10^3	2
106,000	1.06×10^5	3
0.608	6.08×10^{-1}	3
0.0038	3.8×10^{-3}	2
10.05	1.005×10^1	4
762.20	7.6220×10^2	5
0.0320	3.20×10^{-2}	3
0.0000460	4.60×10^{-5}	3
5,000	5×10^3	1
0.0001	1×10^{-4}	1
0.0043	4.3×10^{-3}	2
10,000,000	1×10^7	1
67,500	6.75×10^4	3
0.0620	6.20×10^{-2}	3
0.00750	7.50×10^{-3}	3
400.0	4.000×10^2	4

When approximations (examples: numbers that are measured during a science lab) are multiplied or divided, the answer should have the same number of significant digits as the least accurate factor.

Break for Practice: Calculate the following on your calculator and record your answer in scientific notation with the correct number of significant digits.

1. $(2.32 \times 10^{-6})(4 \times 10^{-5})$

   $9.28 \times 10^{-11} \Rightarrow 9 \times 10^{-11}$

2. $(4.325 \times 10^{5})(9.817 \times 10^{12})$

   $42.458525 \times 10^{17+1} \Rightarrow 4.246 \times 10^{18}$

3. $(4.763 \times 10^{3})(2.48 \times 10^{2})$

   $11.81224 \times 10^{5+1} \Rightarrow 1.18 \times 10^{6}$

4. $\dfrac{2.71 \times 10^{8}}{1.6 \times 10^{-3}}$

   1.69375×10^{11}

   $\Rightarrow 1.7 \times 10^{11}$

5. $\dfrac{7.1 \times 10^{6}}{8.2 \times 10^{1}}$

   $0.8658536 \times 10^{5-1}$

   $\Rightarrow 8.7 \times 10^{4}$

6. $\dfrac{4.2 \times 10^{5}}{(5.7 \times 10^{2})(3.81 \times 10^{10})}$

   $0.193396 \times 10^{-7-1}$

   $\Rightarrow 1.9 \times 10^{-8}$

7. $\left(\dfrac{5.7 \times 10^{4}}{2.96 \times 10^{2}}\right)^{3}$

   $\dfrac{5.7^{3} \times 10^{12}}{2.96^{3} \times 10^{6}}$

   $7.140842 \times 10^{6} \Rightarrow 7.1 \times 10^{6}$

8. $\dfrac{(5.7 \times 10^{2})(3.81 \times 10^{10})}{4.2 \times 10^{5}}$

   $5.170714 \times 10^{7} \Rightarrow 5.2 \times 10^{7}$

Extended Practice: Calculate the following on your calculator and record your answer in scientific notation with the correct number of significant digits.

1. $(1.72 \times 10^{-5})(3.6 \times 10^{-11})$ 6.2×10^{-16}	2. $(8.15 \times 10^{3})(2.0296 \times 10^{-18})$ 1.65×10^{-14}
3. $(8.792 \times 10^{6})(5.31 \times 10^{7})$ 4.67×10^{14}	4. $(3.98 \times 10^{-14})(6.818 \times 10^{19})$ 2.71×10^{6}
5. $(3.29 \times 10^{-3})(1.9532 \times 10^{-4})$ 6.43×10^{-7}	6. $(9.032 \times 10^{-4})(7.91 \times 10^{-17})$ 7.14×10^{-20}

7. $\frac{7.76\times10^{13}}{3.1\times10^{5}}$ 2.5×10^{8}	8. $\frac{8.92\times10^{4}}{2.6\times10^{17}}$ 3.4×10^{-13}
9. $\frac{4.15\times10^{-19}}{5.011\times10^{-4}}$ 8.28×10^{-16}	10. $(1.29 \times 10^{7})^{2}$ 1.66×10^{14}
11. $(6.1 \times 10^{-6})^{2}$ 3.7×10^{-11}	12. $(7.021 \times 10^{5})^{-3}$ 2.889×10^{-18}
13. $\frac{(7.5\times10^{6})(5.0\times10^{-1})}{1.5\times10^{8}}$ 2.5×10^{-2}	14. $\frac{(8.4\times10^{15})(1.5\times10^{-5})}{(4.02\times10^{4})(1.2\times10^{3})}$ 2.6×10^{3}

Simplifying Rational Algebraic Expressions

The rational algebraic expressions that we will be simplifying in this section will include polynomials in the numerator and denominator. The skills that are practiced in this section will help us when we begin graphing in the next section.

Example: Simplify

$$\frac{5x^2+4x-1}{5x^2-10x-15} \longrightarrow 5x^2+5x-x-1$$

$5x(x+1)-1(x+1)$

$(5x-1)(x+1)$

$5(x^2-2x-3)$

$5(x-3)(x+1)$

$$\frac{(5x-1)\cancel{(x+1)}}{5(x-3)\cancel{(x+1)}} = \frac{(5x-1)}{5(x-3)}$$

Steps:

1. Factor the numerator and denominator.
2. Cancel the common factors.

Break for Practice: Simplify

1. $\frac{8x^2+16x}{4x} \rightarrow \frac{\overset{2}{\cancel{8}}\cancel{x}(x+2)}{\cancel{4x}}$

$= 2(x+2)$

2. $\frac{x^2-16}{x^2+7x+12} \rightarrow \frac{\cancel{(x+4)}(x-4)}{(x+3)\cancel{(x+4)}}$

$= \frac{(x-4)}{(x+3)}$

3. $(x^4-5x^3+6x^2)\underbrace{(9x-x^3)^{-1}}_{\text{move to denominator}}$

$\frac{x^4-5x^3+6x^2}{9x-x^3} \rightarrow \frac{x^2(x^2-5x+6)}{x(9-x^2)}$

$\frac{\overset{x}{\cancel{x^2}}(x-2)\overset{-1}{\cancel{(x-3)}}}{\cancel{x}\cancel{(3-x)}(3+x)}$ (opposites cancel and give -1)

$\frac{-x(x-2)}{(3+x)}$

4. $(2-x-3x^2)(9x^2-4)^{-1}$

$\frac{-1(3x^2+x-2)}{9x^2-4} \rightarrow 3x^2+3x-2x-2$

$3x(x+1)-2(x+1)$

$(3x+2)(3x-2)$ $\quad (3x-2)(x+1)$

$\frac{-1\cdot\cancel{(3x-2)}(x+1)}{(3x+2)\cancel{(3x-2)}}$

$\frac{-1\cdot(x+1)}{(3x+2)}$

Extended Practice: Simplify.

1. $\frac{5x^2-15x}{10x^2} = \frac{(x-3)}{2x}$	2. $\frac{3t^4-9t^3}{6t^2} = \frac{t(t-3)}{2}$
3. $\frac{u^2-u-2}{u^2+u} = \frac{u-2}{u}$	4. $\frac{z^3-4z}{z^2-4z+4} = \frac{z(z+2)}{(z-2)}$

5. $(p-q)(q-p)^{-1} = -1$	6. $(r^2-rs)(r^2-s^2)^{-1} = \frac{r}{r+s}$
7. $\frac{s^2-t^2}{(t-s)^2} = \frac{-1(s+t)}{(t-s)}$	8. $\frac{(a-x)^2}{x^2-a^2} = \frac{-1\cdot(a-x)}{(x+a)}$
9. $\frac{x^2-5x+6}{x^2-7x+12} = \frac{(x-2)}{(x-4)}$	10. $\frac{2t^2+5t-3}{2t^2+7t+3} = \frac{(2t-1)}{(2t+1)}$

Graphing Rational Functions

This section will investigate the graphs of rational functions. Rational functions can include unique features such
as holes and asymptotes. Since graphing calculators do not always show these features well, it is important to know how to algebraically find these features.

Example: Graph the following function and identify the locations of all holes and/or vertical asymptotes.

$$f(x) = \frac{x^2 - 1}{x^2 + 2x - 3}$$

Steps:

1. Simplify the function by factoring the top **and** bottom as much as possible.
2. Use the factors of the denominator to identify the excluded values. (Set the factors in the denominator equal zero.)
3. Reduce function
4. Find the vertical asymptotes and/or holes by substituting the excluded values into the reduced function.
   a) If you get $\frac{\#}{0}$, then it's a vertical asymptote. → vertical line (dashed): $x =$ ___
   b) If you get $\frac{\#}{\#}$, then it's a hole. → open point: ordered pair (x, y)
5. Complete the graph with the graphing calculator.

① $f(x) = \dfrac{(\cancel{x-1})(x+1)}{\underbrace{(x+3)(\cancel{x-1})}}$

② $x+3=0 \qquad x-1=0$

$x \neq -3 \qquad x \neq 1$

$\boxed{\text{Excluded values: } x \neq -3 \text{ or } 1}$

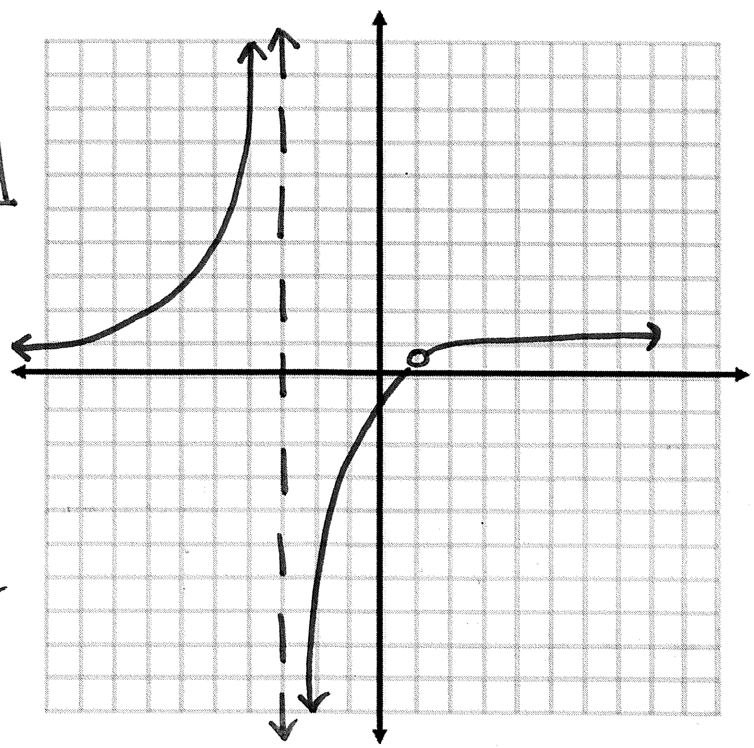
③ $f(x) = \dfrac{(x+1)}{(x+3)}$

④ $f(\overset{x}{-3}) = \dfrac{-3+1}{-3+3} = \dfrac{-2}{0} \rightarrow \dfrac{\#}{0}$ ← y d.n.e

$\boxed{\text{Asymptote @ } x = -3}$

$f(\overset{x}{1}) = \dfrac{1+1}{1+3} = \dfrac{2}{4} = \overset{y}{\dfrac{1}{2}} \rightarrow \dfrac{\#}{\#}$

Hole @ $(\underset{x}{1}, \underset{y}{1/2})$



⑤ Put original f^{n} into calculator

$\boxed{y=}$ $(x^2-1)\boxed{\div}(x^2+2x-3)$

Break for Practice: Graph the following function and identify the locations of all holes and/or vertical asymptotes.

1. $f(x) = \frac{x^2-2x-8}{x^2-x-12}$

① $f(x) = \frac{(x-4)(x+2)}{(x-4)(x+3)}$

② $x-4=0 \quad x+3=0$

$x \neq 4 \quad x \neq -3$

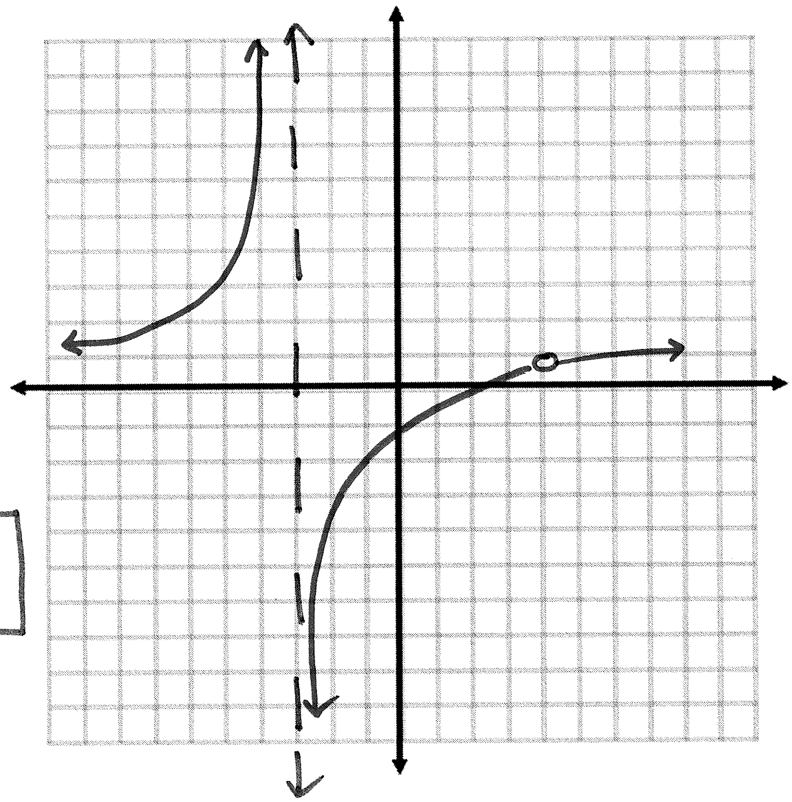
$\boxed{\text{Excluded Values: } x \neq 4, -3}$

③ $f(x) = \frac{(x+2)}{(x+3)}$

④ $f(4) = \frac{4+2}{4+3} = \frac{6}{7}$ $\boxed{\text{Hole @ } (4, 6/7)}$

$f(-3) = \frac{-3+2}{-3+3} = \frac{-1}{0}$ $\boxed{\text{V.A @ } x=-3}$

y d.n.e



2. $f(x) = \frac{5}{x^2-2x-8}$

① $f(x) = \frac{5}{(x-4)(x+2)}$

② $x-4=0 \quad x+2=0$

$x \neq 4 \quad x \neq -2$

$\boxed{\text{Excluded Values: } x \neq 4, -2}$

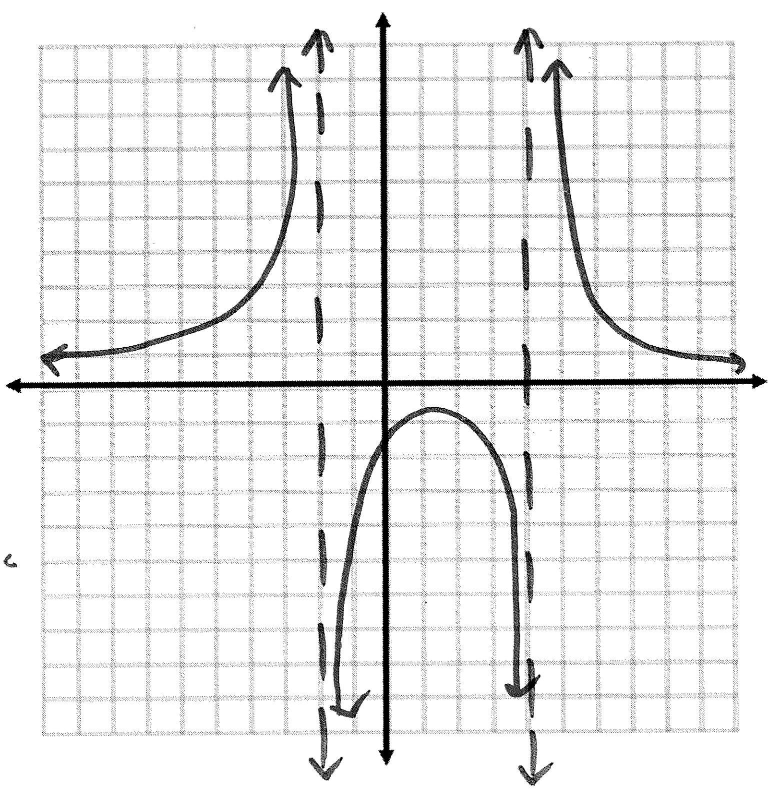
③ $f(x) = \frac{5}{(x-4)(x+2)}$ *cannot be reduced*

④ $f(4) = \frac{5}{(4-4)(4+2)} = \frac{5}{0}$ y d.n.e

$\boxed{\text{V.A. @ } x = 4}$

$f(-2) = \frac{5}{(-2-4)(-2+2)} = \frac{5}{0} \leftarrow$ y d.n.e

$\boxed{\text{V.A. @ } x = -2}$



3. $f(x) = \frac{x^2+6x-7}{x-1}$

① $f(x) = \frac{(x+7)(x-1)}{(x-1)}$

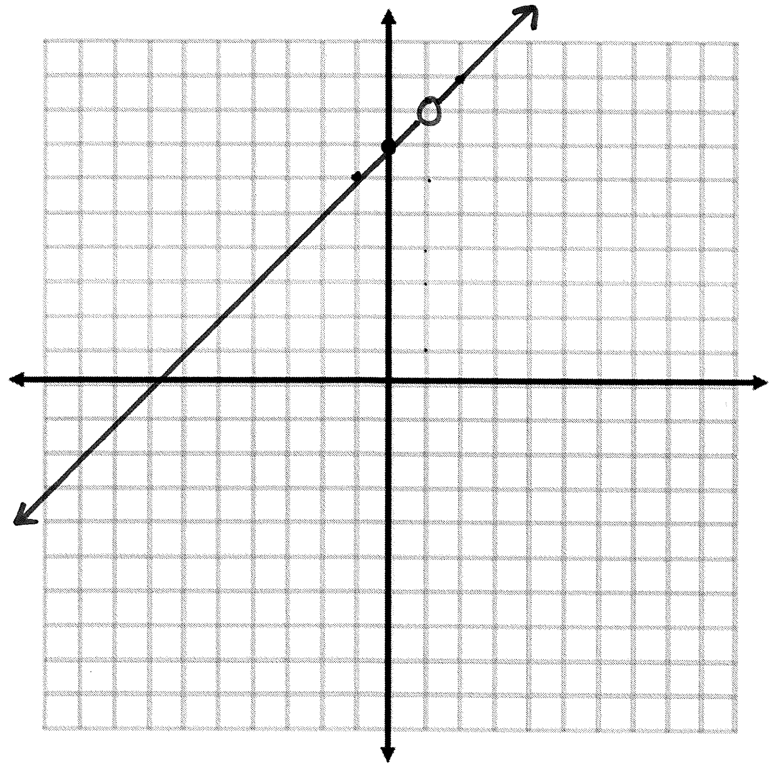
② $x-1=0$
$x \neq 1$

$\boxed{\text{Excluded Value } x \neq 1}$

③ $f(x) = x+7$

④ $f(1) = 1+7$
$= 8$

$\boxed{\text{Hole @ } (1,8)}$

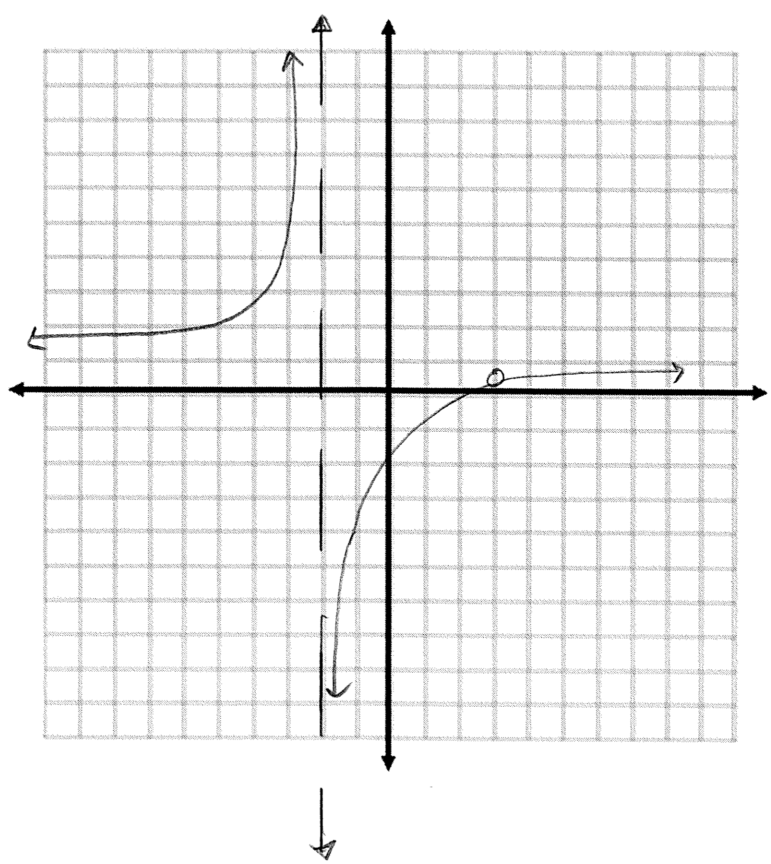


Extended Practice: Graph the following function and identify the locations of all holes and/or vertical asymptotes.

1. $f(x) = \frac{x^2-4x+3}{x^2-x-6}$

$\boxed{\text{Excluded Values: } x \neq -2, -3}$

Vertical Asymptote @ $x = -2$
Hole @ $(3, \frac{2}{5})$

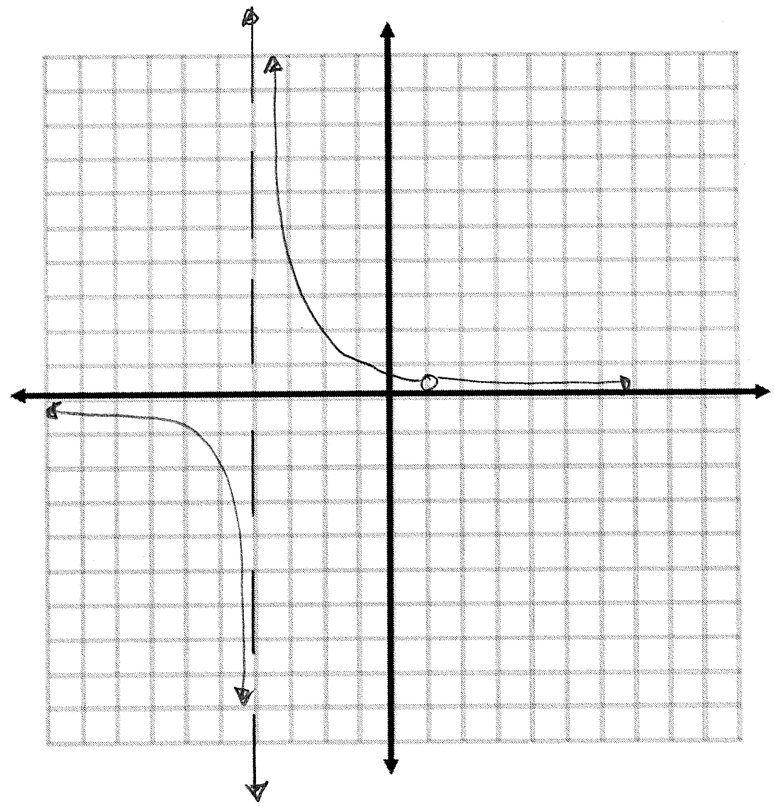


2. $f(x) = \frac{x-1}{x^2+3x-4}$

Excluded Values: $x \neq 1, -4$

Vertical Asymptote @ $x = -4$

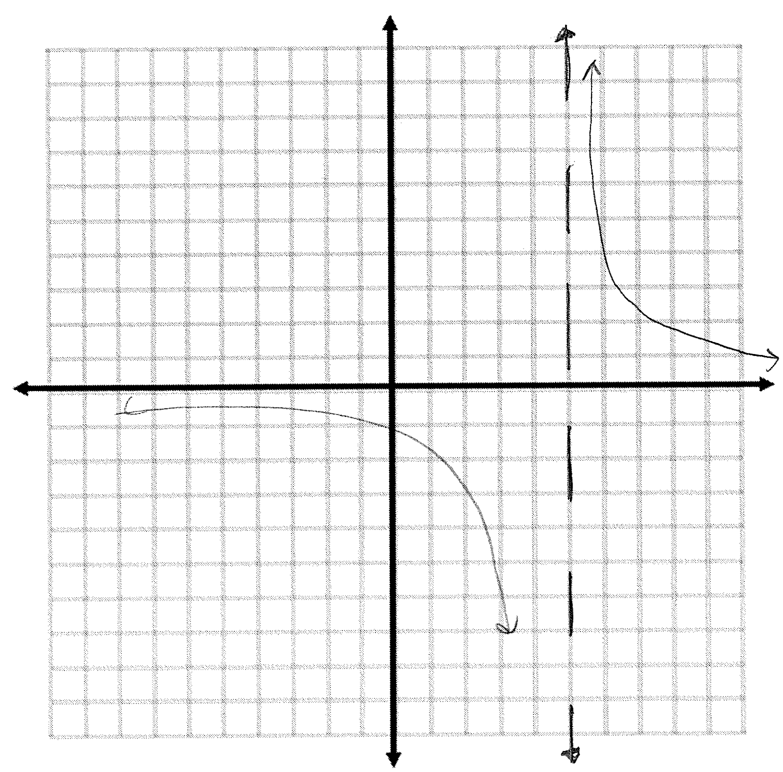
Hole @ $(1, \frac{1}{5})$



3. $f(x) = \frac{4}{x-5}$

Excluded Values: $x \neq 5$

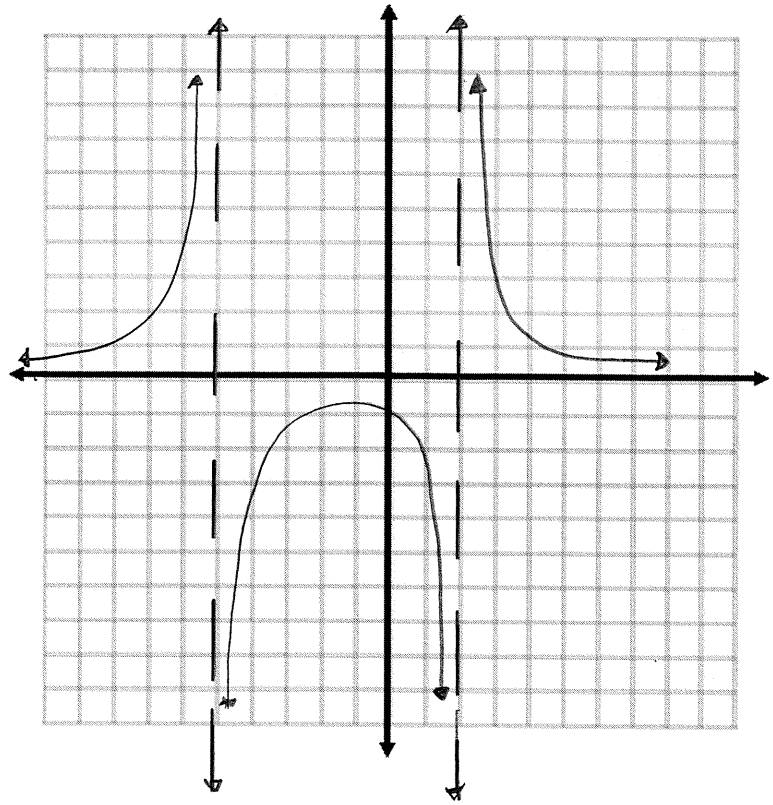
Vertical Asymptote @ $x = 5$



4. $f(x) = \frac{2}{x^2+3x-10}$

Excluded Values: $x \neq 2, -5$

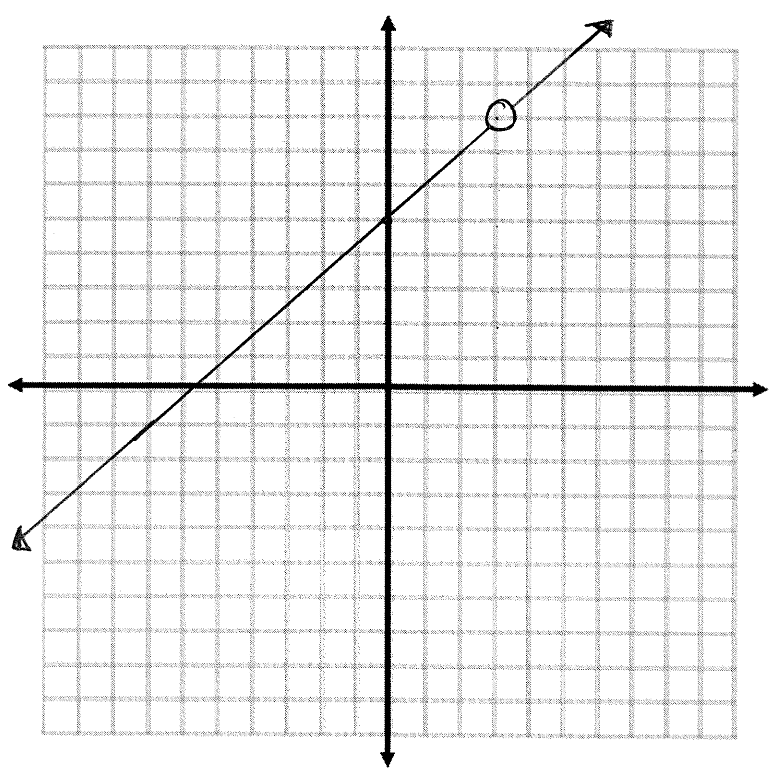
Vertical Asymptote @ $x = 2$
Vertical Asymptote @ $x = -5$



5. $f(x) = \frac{x^2+2x-15}{x-3}$

Excluded Value: $x \neq 3$

Hole @ $(3, 8)$



Products and Quotients of Rational Algebraic Expressions

In this section we will learn how to multiply and divide rational expressions. We will begin by reviewing the techniques with simple numerical fractions.

Review: Simplify

1. $\frac{\overset{3}{\cancel{15}}}{\underset{4}{\cancel{24}}} \cdot \frac{\overset{3}{\cancel{18}}}{\underset{5}{\cancel{25}}} \Rightarrow \frac{9}{20}$

OR $\frac{15}{24} \cdot \frac{18}{25} = \frac{27\cancel{0}}{60\cancel{0}}\ (\div 3)\ \frac{9}{20}$

2. $\frac{12}{32} \div \frac{18}{24} \Rightarrow \frac{\overset{2}{\cancel{12}}}{\underset{4}{\cancel{32}}} \cdot \frac{\overset{3}{\cancel{24}}}{\underset{3}{\cancel{18}}} \Rightarrow \frac{6}{12} = \frac{1}{2}$

Leave it

Change it

Flip it

OR $\frac{12}{32} \cdot \frac{24}{18} = \frac{288}{576} = \frac{1}{2}$

Now it is time to try the same thing with rational algebraic expressions. The same technique applies, but polynomials should be factored before any reducing takes place.

Break for Practice: Simplify

1. $\frac{1}{x+2} \cdot \frac{\overset{(x-2)(x+2)}{x^2-4}}{1} \Rightarrow \frac{1}{(x+2)} \cdot \frac{(x-2)(x+2)}{1} = \frac{(x-2)\cancel{(x+2)}}{\cancel{(x+2)}} \boxed{= x-2}$

2. $\frac{x+3}{x-5} \div \frac{\overset{\text{Flip}}{1}}{\underset{(x-5)(x+3)}{x^2-2x-15}} \Rightarrow \frac{x+3}{x-5} \cdot \frac{(x-5)(x+3)}{1} = \frac{(x+3)\cancel{(x-5)}(x+3)}{\cancel{(x-5)}} \boxed{= (x+3)^2}$

3. $\frac{x^2+3x-10}{x^2-7x+6} \cdot \frac{x^2+2x-3}{x^2+x-6} \Rightarrow \frac{(x+5)\cancel{(x-2)}}{(x-6)\cancel{(x-1)}} \cdot \frac{\cancel{(x+3)}\cancel{(x-1)}}{\cancel{(x+3)}\cancel{(x-2)}} \boxed{= \frac{x+5}{x+6}}$

4. $\frac{(x-3)^2}{9+3x} \div \frac{\overset{\text{Flip}}{9-x^2}}{x^2+3x} \Rightarrow \frac{(x-3)\cancel{(x-3)}}{3\cancel{(3+x)}} \cdot \frac{x(x+3)}{\underset{-1}{\cancel{(3-x)}}(3+x)} = \boxed{-\frac{x(x-3)}{3(3+x)}}$

5. $\frac{x^2-x}{1-x} \cdot \frac{x+2}{x^2-2x} \Rightarrow \frac{\cancel{x}\cancel{(x-1)}}{\underset{-1}{\cancel{(1-x)}}} \cdot \frac{(x+2)}{\cancel{x}(x-2)} = \boxed{-\frac{x+2}{x-2}}$

Extended Practice: Simplify.

1. $\frac{2}{x-2} \cdot \frac{x^2-4}{4}$ $= \frac{x+2}{2}$

2. $\frac{x^2+7x+12}{12} \cdot \frac{4}{x+4}$ $= \frac{x+3}{3}$

3. $\frac{x^2-64}{x^2-16} \div \frac{x+8}{x+4}$ $= \frac{x-8}{x-4}$

4. $\frac{x^2+6x}{6} \cdot \frac{x^2+6}{x^3+6x^2}$ $= \frac{x^2+6}{6x}$

5. $\frac{x^2-4}{2x-4} \cdot \frac{2}{x+2}$ $= 1$

6. $\frac{x^2-9}{x^2+x} \div \frac{3-x}{x^2-1}$ $= -\frac{(x+3)(x-1)}{x}$

Sums and Differences of Rational Algebraic Expressions

This section will review how to add and subtract numerical fractions, and then extend the same techniques to rational algebraic expressions.

① Find a common deminator (LCM)
② What you do to the bottom - you do for the top
③ Add / Subtract
④ Simplify if you can

Review: Simplify

$$\frac{(3)\,5}{(3)\,12}+\frac{7\,(2)}{18\,(2)}=\frac{15}{36}+\frac{14}{36}=\frac{29}{36}$$

LCM: 36

The same technique works for subtracting fractions.

The same technique also works for rational algebraic expressions.

Break for Practice: Simplify

1. $\frac{5}{6k}-\frac{4\,(2)}{3k\,(2)}=\frac{5}{6k}-\frac{8}{6k}$

LCM: 6K

$$=\frac{-3}{6k}$$

$$=-\frac{1}{2k}$$

2. $\frac{(2)\,a-1}{(2)\,a^2}+\frac{a+1\,(a)}{2a\,(a)}=\frac{2a-2}{2a^2}+\frac{2a^2+a}{2a^2}$

LCM: $2a^2$

$$=\frac{2a^2+3a-2}{2a^2}$$

3. $\frac{(x+1)\,3}{(x+1)\,x-1}+\frac{2\,(x-1)}{x+1\,(x-1)}=\frac{3x+3}{(x+1)(x-1)}+\frac{2x-2}{(x+1)(x-1)}$

LCM: $(x-1)(x+1)$

$$=\frac{5x+1}{(x+1)(x-1)}$$

4. $\frac{(x+2)\,x-1}{(x+2)\,x+2}-\frac{x^2-5x-2}{(x+2)(x+2)\,x^2+4x+4}=\frac{(x+2)(x-1)}{(x+2)(x+2)}-\frac{(x^2-5x-2)}{(x+2)(x+2)}$

LCM: $(x+2)(x+2)$ $=\frac{\cancel{x^2}-x+2x-2-\cancel{x^2}+5x+2}{(x+2)(x+2)}$

$$=\frac{6x}{(x+2)(x+2)}$$

Extended Practice: Simplify

1. $\frac{t+2}{3}+\frac{t-4}{6}=\frac{t}{2}$	2. $\frac{z-1}{z}+\frac{z+1}{z^2}=\frac{z^2+1}{z^2}$

3. $\frac{x+2}{x^2}+\frac{x-2}{2x} = \frac{x^2+4}{2x^2}$

4. $\frac{t-4}{2t}-\frac{t-6}{3t} = \frac{1}{6}$

5. $\frac{1}{x+1}+\frac{1}{x-1} = \frac{2x}{(x+1)(x-1)}$

6. $\frac{1}{1-2x}-\frac{2}{1-4x^2} = \frac{-1}{1+2x}$

7. $\frac{3}{x^2-5x+6}-\frac{2}{x^2-4} = \frac{x+12}{(x+2)(x-2)(x-3)}$

Equations with Fractional Coefficients

In this section we will use some of what we have learned in this unit to solve equations and inequalities that have fractional coefficients. The key to solving these problems is to **<u>clear the fractions first by multiplying through by the least common denominator.</u>** Yay, no more fractions!! ☺

Steps for Solving Equations with Fractional Coefficients: Find LCM of the denominators → LCD Multiply **<u>BOTH</u>** sides of the equation/inequality by the LCD (this will clear the denominators) Solve (this may require simplifying, solving for a variable, factoring, etc)

Break for Practice: Solve

1. $\frac{4t}{3}+\frac{3t}{10}=\frac{7}{5}$

$\overset{10}{\cancel{30}}\left(\frac{4t}{\cancel{3}}\right)+\overset{3}{\cancel{30}}\left(\frac{3t}{\cancel{10}}\right)=\overset{6}{\cancel{30}}\left(\frac{7}{\cancel{5}}\right) \Rightarrow 40t+9t=42$

$\frac{49t}{49}=\frac{42}{49}$

$t=\frac{6}{7}$

LCM: 30

2. $\frac{3r-4}{5}-\frac{2r+1}{4}=-\frac{1}{2}$

$\overset{4}{\cancel{20}}\left(\frac{3r-4}{\cancel{5}}\right)-\overset{5}{\cancel{20}}\left(\frac{2r+1}{\cancel{4}}\right)=\overset{10}{\cancel{20}}\left(-\frac{1}{\cancel{2}}\right) \Rightarrow 4(3r-4)-5(2r+1)=-10$

$12r-16-10r-5=-10$

$2r-21=-10$

$\quad +21 \quad +21$

$\frac{2r}{2}=\frac{11}{2}$

$r=11/2$

LCM: 20

3. $\frac{y+3}{2}+\frac{3}{5}\geq\frac{y+1}{10}$

$\overset{5}{\cancel{10}}\left(\frac{y+3}{\cancel{2}}\right)+\overset{2}{\cancel{10}}\left(\frac{3}{\cancel{5}}\right)\geq\overset{1}{\cancel{10}}\left(\frac{y+1}{\cancel{10}}\right) \Rightarrow 5(y+3)+2(3)\geq y+1$

$5y+15+6\geq y+1$

$5y+21\geq y+1$

$-y \quad -y$

$4y+21\geq 1$

$-21 \quad -21$

$\frac{4y}{4}\geq\frac{-20}{4}$

$y\geq -5$

LCM: (10)

4. How many liters of pure acid must be added to 5 liters of a solution that is 20% acid to make a solution that is 60% acid?

	mL of Solution x	%	= mL of acid
20% Solution	5	20% → 0.20	0.20 x
Pure Acid	x	100% → 1	x
60% Solution	x + 5	60% → 0.60	0.6(x + 5)

$0.20x + x = 0.6(x+5)$

$1.2x = 0.6x +$

$-0.6x \quad -0.6x$

$\frac{0.6x}{0.6} = \frac{3}{0.6}$

$x = 5$ Liters

5. Pump A can unload the *Lunar Petro* in 30 hours and pump B can unload it in 24 hours. Because of an approaching storm, both pumps were used. How long did they take to empty the ship?

1/30   1/24

Pump A → takes 30 hours for 1 job.
1/30 of job completed per hour

Pump B → takes 24 hours for 1 job
1/24 of job completed per hour

t = time taken

$240\left(\frac{1}{30}t\right) + 240\left(\frac{1}{24}t\right) = (1)\,240 \Rightarrow 8t + 10t = 240$

LCM: 240

$\frac{18t}{18} = \frac{240}{18}$

$t = 13\frac{1}{3}$ hours

Extended Practice: Solve.

1. $\frac{x}{9} + \frac{1}{6} = \frac{2}{3}$ $x = 9/2$

LCM: 18

2. $\frac{3u}{5} - \frac{5}{6} = \frac{u}{10}$ $u = \frac{5}{3}$

LCM: 30

3. $\frac{s-2}{2} - \frac{s-1}{5} = \frac{1}{4}$ $s = \frac{7}{2}$

LCM: 20

4. $\frac{z}{4} - \frac{z-1}{6} \leq \frac{5}{12}$ $z \leq 3$

LCM: 12

5. $\frac{r-2}{8} < \frac{3r+1}{6} + \frac{1}{3}$ $r > -2$

LCM: 24

6. $\frac{t^2}{6} - \frac{t}{2} - \frac{2}{3} = 0$ $t = 4,\ t = -1$

LCM: 6

7. $\frac{t(t-1)}{3} = \frac{t+1}{2}$ $t = -\frac{1}{2},\ t = 3$

LCM: 6

8. An old conveyor belt takes 21 hours to move one day's coal output from the mine to a rail line. A new belt can do it in 15 hours. How long does it take when both are used at the same time?

t = 8.75 hours or t = 8 hours and 45 mins

9. How much pure antifreeze must be added to 12 liters of a 40% solution of antifreeze to obtain a 60% solution?

6 Liters of pure antifreeze would need to be added

Solving Fractional Equations

Closely related to the problems that were solved in the previous section, are fractional equations. These equations include variables in the denominator which means that you have to be alert to extraneous (extra solutions that don't work in the original equation) solutions. The key is to remember that the denominators can never equal zero.

Steps for Solving Fractional Equations:

1. Find your restrictions (excluded values)
2. Find the LCD → Remember this is finding the LCM of all the denominators in the equation
3. Multiply BOTH sides of the equation → make sure to distribute to each term
4. Simplify and Solve
5. Check your answer against the restrictions!!

Break for Practice: Solve

1. $\frac{3}{y} - \frac{1}{2y} = \frac{5}{4}$

$4y\left(\frac{3}{y}\right) - \cancel{4y}^{2}\left(\frac{1}{\cancel{2y}}\right) = \cancel{4}y\left(\frac{5}{\cancel{4}}\right) \Rightarrow 12 - 2 = 5y$

$\frac{10}{5} = \frac{5y}{5}$

$2 = y$

LCM: $4y$

Restrictions: $\frac{4y}{4} = \frac{0}{4}$

$y \neq 0$

2. $\frac{12}{n} = \frac{12}{n+1} + 1$

$(\cancel{n})(n+1)\left(\frac{12}{\cancel{n}}\right) = (n)(\cancel{n+1})\left(\frac{12}{\cancel{n+1}}\right) + (n)(n+1)\,1$

LCM: $n(n+1)$

Restrictions: $n \neq 0$ $\quad n+1=0$

$n \neq -1$

$12(n+1) = 12n + (n)(n+1)$

$12n + 12 = 12n + n^2 + n$

$-12n \quad -12 \qquad -12n \qquad -12$

$0 = n^2 + n - 12$

$0 = (n+4)(n-3)$

$n+4=0 \qquad n-3=0$

$n=-4 \qquad n=3$

3. $\frac{7}{x-3} - \frac{3}{x-4} = \frac{1}{2}$

$(\cancel{x-3})(x-4)2\left(\frac{7}{\cancel{x-3}}\right) - (x-3)(\cancel{x-4})2\left(\frac{3}{\cancel{x-4}}\right) = (x-3)(x-4)\cancel{2}\left(\frac{1}{\cancel{2}}\right)$

LCM: $(x-3)(x-4)2$

$7(2x-8) - 3(2x-6) = x^2 - 4x - 3x + 12$

$14x - 56 - 6x + 18 = x^2 - 7x + 12$

Restrictions: $x-3=0 \qquad x-4=0$

$x \neq 3 \qquad x \neq 4$

$8x - 38 = x^2 - 7x + 12$

$-8x \quad +38 \qquad -8x \quad +38$

$0 = x^2 - 15x + 50$

$0 = (x-10)(x-5)$

$x-10=0 \qquad x-5=0$

$x=10 \qquad x=5$

$\frac{7}{7} - \frac{3}{6} \qquad \frac{7}{2} - \frac{3}{1}$

4. $\frac{12}{x^2-4} - \frac{3}{x-2} = -1$

$(x-2)(x+2)$

LCM: $(x-2)(x+2)$

Restrictions: $x-2=0$ $\quad x+2=0$

$x \neq 2 \quad x \neq -2$

$(\cancel{x+2})(\cancel{x-2})\left(\frac{12}{(\cancel{x-2})(\cancel{x+2})}\right) - (x+2)(\cancel{x-2})\left(\frac{3}{(\cancel{x-2})}\right) = -1(x+2)(x-2)$

$12 - 3(x+2) = -1(x+2)(x-2)$

$12 - 3x - 6 = -(x^2 - \cancel{2x} + \cancel{2x} - 4)$

$6 - 3x = -(x^2 - 4)$

$6 - 3x = -x^2 + 4$

$+x^2 \; -4 \qquad +x^2 \; -4$

$x^2 - 3x + 2 = 0$

$(x-2)(x-1) = 0$

$x-2=0 \quad x-1=0$

$\cancel{x=2} \quad x=1$

Restricted Value

5. One pump can empty the town swimming pool in 7 hours less time than a smaller second pump can. Together they can empty the pool in 12 hours. How much time would it take the larger pump alone to empty it?

Fast pump: $x-7$ hrs (how long it takes to complete a job alone)

$\frac{1}{x-7}$ how much of a job it completes in 1 hr.

Slow Pump: x hrs. (how long it takes to complete a job alone)

$\frac{1}{x}$ how much of a job it completes in 1 hr.

$\frac{1}{x-7}(12) + \frac{1}{x}(12) = 1$

$\frac{12}{x-7} + \frac{12}{x} = 1 \Rightarrow (x)(\cancel{x-7})\left(\frac{12}{\cancel{x-7}}\right) + (\cancel{x})(x-7)\left(\frac{12}{\cancel{x}}\right) = (x)(x-7)1$

LCM: $(x)(x-7)$

Restrictions: $x \neq 0$

$x-7=0$

$x \neq 7$

$12x + 12(x-7) = x(x-7)$

$12x + 12x - 84 = x^2 - 7x$

$24x - 84 = x^2 - 7x$

$-24x \; +84 \qquad -24x \; +84$

$0 = x^2 - 31x + 84$

$0 = (x-3)(x-28)$

$x-3=0 \quad x-28=0$

$x=3 \quad x=28$

(try 3 and 28)

$x, x-7 \qquad x, x-7$

$3, 3-7 \qquad 28, 28-7$

$\cancel{3, -4} \qquad 28, 21$

$\boxed{\text{It would take 21 hours}}$

Extended Practice: Solve.

1. $\frac{3}{t} - \frac{1}{3t} = \frac{2}{3}$ $\quad t = 4$

LCM: $3t$

Restrictions: $t \neq 0$

2. $\frac{1}{x} = \frac{2}{x-3}$ $\quad x = -3$

LCM: $(x)(x-3)$

Restrictions: $x \neq 0, 3$

3. $\frac{2}{s+3} - \frac{1}{s-3} = 0$ $S = 9$

LCM: $(s+3)(s-3)$

Restrictions: $S \neq -3, 3$

4. $\frac{x}{x+3} + \frac{1}{x-3} = 1$ $X = 6$

LCM: $(x+3)(x-3)$

Restrictions: $X \neq -3, 3$

5. $\frac{1}{y-2} + \frac{1}{y+2} = \frac{4}{y^2-4}$ No Solution.

LCM: $(y-2)(y+2)$

Restrictions: $y \neq 2, -2$

6. $\frac{3}{x+1} - \frac{1}{x-2} = \frac{1}{x^2-x-2}$ $X = 4$

LCM: $(x+1)(x-2)$

Restrictions: $X \neq -1, 2$

7. A town's old street sweeper can clean the streets in 60 hours. The old sweeper together with a new sweeper can clean the streets in 15 hours. How long would it take the new sweeper to do the job alone?

$t = 20$hours