Algebra II
Unit 4: Polynomials

<u>Priority Standard:</u> A-SSE.3a- Factor a quadratic expression to reveal the zeros of the function it defines.

Unit "I can" statements:

1. I can simplify, add, subtract, and identify the degree of polynomials.
2. I can use the laws of exponents to multiply a monomial and polynomial.
3. I can multiply polynomials.
4. I can use prime factorization to identify the GCF and LCM of integers and monomials.
5. I can factor polynomials by applying a variety of strategies.
6. I can solve polynomial equations.
7. I can solve applications by using polynomial equations.
8. I can solve polynomial inequalities.

Common Core State Standards that are addressed in this unit include: A.CED.2a, A.CED.3a, A.SSE.1a, A.SSE.1b, A.SSE.2a, F.IF.8
For more information see www.corestandards.org/Math/

Introduction to Polynomials

In this unit we will focus on a special type of expression called a polynomial. We will learn how to simplify, operate with, and factor polynomials. We will also investigate applications.

Definition:

Polynomials – expressions that involve only the operations of addition, subtraction, and multiplication on variables.

- Exponents have to be positive whole numbers.
- Terms are separated by addition and subtraction.

Break for Practice: Identify which of the following are polynomials. For those that are polynomials, state the number of terms. For those that are not polynomials, explain why.

1. $4x^2 + 5x - 7$
2. $3x - 2$
3. $\frac{2+x}{4x}$
4. $\frac{4+x}{3}$
5. $7x^2y^5$
6. $\sqrt{7x}$
7. $\sqrt{7}x$
8. $|x^3 - 2|$
9. $5x^2y^3 - 2xy^5 - 7$
10. 8

Definition:

Factors – the parts of a term that are multiplied together to form the term.

Example: Consider $7x^4$

- In the term $7x^4$, there are __________ factors. They are ____________________________.
- In the term $7x^4$, 7 is called the ______________________, x is called the ________________ , and 4 is called the ______________________.

Definition:

Degree of a Polynomial – is the maximum number of variables that appear as factors in any one term of the polynomial.

Rule: *Add the exponents on the variables for each individual term. Whichever term has the highest value, gives the degree of the polynomial.*

Example: What is the degree of $7x^2y - 4xy^3z + 6x^2y^2z^3$?

There are _____ terms. The first term is __________, and its degree is __________. The second term is __________ , and its degree is __________. The third term is _______________ , and its degree is _________. The highest degree value is __________, so that is the degree of the polynomial.

Break for Practice: State the degree of each of the following polynomials.

1. $5x^4 - 3x^3 + 7x + 2$
2. $4x^3y^2 - 7x^3y^3 + 6xy^2$
3. $-2x^2y^3z^4$
4. $4x^3 - 3x^2y - 7x + 2$
5. $5x^2y^3 - 2xy^5 - 7$
6. $3x - 2x^3 - 9 + 2x^3 + 2 - 5x^2$

Definition:

Like Terms – like terms have identical variable parts.

Example: $7x^2y$ and _____________ are like terms because the variable combination are **exactly** the same.

To add or subtract terms together, they **MUST** be like terms.

Simplifying Polynomials means that all ______________________________ are combined, and we usually arrange the terms in order of _____________________________ degrees of one of the variables.

Example: Simplify $7xy^3 + 3x^2y - 2x^3 + 6xy^3 - x^2y - 3x^3$

Break for Practice: Simplify each polynomial.

1. $x^4 - x^3 + 3x^4 - 2x^3 + 3x^2$
2. $4x^3y^2 + 2x^2y - 6x^3y^2 - 4xy + 7xy + 5x^2y$
3. $x^3 + 4x^2 - 3x^3 + x - 7x^2 + 8$
4. $2xy - 3yz + 4xz - 2xy + 3yz - 4xz$

Polynomials can be added and subtracted by combining like terms. In the case of subtraction, it helps to first distribute a negative one.

Addition Example: $(x^2 - 2x + 1) + (3x^2 + 5x + 4)$

Subtraction Example: $(x^2 - 2x + 1) - (3x^2 + 5x + 4)$

Extended Practice: Simplify each of the following by performing the indicated addition or subtraction.

1. $(3x^2 - 4x + 5) + (2x^2 - 3x - 1)$
2. $(3x^2 - 4x + 5) - (2x^2 - 3x - 1)$
3. $(x^3 - 2x^2) + (3x^2 + 5x + 1)$
4. $(x^3 - 2x^2) - (3x^2 + 5x + 1)$
5. $(-5x^2 - 2x + 7) + (-2x^2 - 3x + 4)$
6. $(-5x^2 - 2x + 7) - (-2x^2 - 3x + 4)$
7. $3(x^2 - 3x + 4) + 2(3x^2 - 4x - 1)$
8. $3(x^2 - 3x + 4) - 2(3x^2 - 4x - 1)$

Using Laws of Exponents

The next skill that needs to be learned when dealing with polynomials is how to simplify expressions by applying the laws of exponents. The exponent laws that we will use in this section should be familiar to you from Algebra I, but it is wise to review.

Example	Law of Exponents
$x^2 \cdot x^3$	
$(x^2)^3$	
$(x \cdot y)^3$	

***Reminder**: $x = x^1$

***Note**: It is important to understand the difference between these two similar looking expressions. Simplify each.

-4^2 $\quad\quad\quad\quad$ $(-4)^2$

If you want to square a negative number on your calculator, you must put that negative number in parentheses.

Break for Practice: Simplify

1. $c^4 \cdot c^2$
2. $(mn^2)^4$
3. $(-x^5)^2$
4. $5x^3 \cdot 2x^2$
5. $(x^2y^3)^5$
6. $(6m^4n^3)(2mn)$
7. $(6c^2d^4e^5)^2$
8. $(2r^2)^3(3r)^2$

9. $3x^2(5x^2+3x-2)$

10. $a^2b^3(3a^3-4a^2b-2ab^2+2)$

Extended Practice: Simplify

1. $5r^2 \cdot r^4$	2. $(-t^3)^4$
3. $(4p^2q)(p^2q^3)$	4. $(r^2s)(-3rs^3)(2rs)$
5. $(2c^2d^3)^3$	6. $(-x^2yz^3)^4$
7. $(-c)^2(-c^4)$	8. $(2x^2y^3)^3(3x^3y)^2$
9. $x^2(x-2x^2+3x^3)$	10. $p^2q^3(p^2-4q)$
11. $t^4 \cdot t^{k-4}$	12. $y^{p+2} \cdot y^p \cdot y^{p-2}$
13. $x^3(x^{2k-1})^3$	14. $rs^2(r^2-2rs-s^2)$

Multiplying Polynomials

Now that we have reviewed several laws of exponents and used them to multiply monomials, we will extend the ideas to multiplying polynomials.

Do you remember how to multiply and simplify something like this?

$(2x + 4)(3x - 7)$

Break for Practice: Simplify.

1. $(a + 2)(3a - 5)$
2. $(7y + z)(2y + 5z)$
3. $(y + 6)^2$
4. $(2a - 3)^2$

For multiplying larger polynomials, there are a couple of different techniques that can be used.

Example: Multiply $(2x - 1)(x^2 - 3x + 5)$

<u>Horizontal Method</u>

<u>Vertical Method</u>

Break for Practice: Simplify

$(x^2 + 3)(x^4 + 2x^2 - 1)$

Extended Practice: Simplify

1. $(4z+3)(3z-4)$	2. $(4k-5)^2$
3. $(7t+2)(2t-1)$	4. $(9-5t)(5t-9)$
5. $(2p+3q)(3p-2q)$	6. $(p^2-2q^2)(p^2+2q^2)$
7. $(2x^2-5)^2$	8. $t(t-2)(t+1)$
9. $mn(m-n)(m-2n)$	10. $(x+2)(x^2+3x-5)$
11. $(3-k^2)(2-k^2-k^4)$	12. $(a+2b)(a^3-2a^2b-b^3)$

Using Prime Factorization

For the next few sections we will be studying factoring. Before we learn techniques for factoring polynomials, we will work with monomials. In this section, we will concentrate on prime factors, greatest common factors, and least common multiples.

A **<u>factor</u>** is a number that divides into a given number.

Example: List all of the factors of 24. ____

A **<u>prime</u>** is a number whose only factors are ____________________________.

Example: List the first ten primes. ____

<u>Greatest Common Factor (GCF)</u>: When working with two or more numbers or expressions, the GCF is the largest number/ expression that is a factor of all of the given numbers/expressions.

Example: Find the GCF of 18 and 24. __________

<u>Note</u>: When you are reducing a fraction, you are looking for the GCF.

<u>Least Common Multiple (LCM):</u> When working with two or more numbers or expressions, the LCM is the smallest positive number/expression that has each of the given number/expressions as factors.

Example: Find the LCM of 18 and 24. __________

<u>Note</u>: When you are looking for a common denominator, you are actually looking for the LCM.

One method for finding the GCF or LCM involves using prime factorization. Let's review this.

Example: Write the prime factorizations for 54 and 90.

Rules for finding GCF and LCM:

1. To find the GCF, take the product of each common factor raised to its lowest power.
2. To find the LCM, take the product of every factor raised to its highest power.

Example: Find the GCF and LCM of 54 and 90.

Break for Practice: Find the GCF and LCM of each of the following monomials.

1. 60 and 72

2. $42xy$ and $-70yz$

3. $10rs$, $12rs^4t$, and $14r^3s^2t^2$

Extended Practice: Find the GCF and LCM for each of the following monomials.

1. 45 and 75	2. 315 and -525

3. 30, 35, 36, and 42	4. $49x^3$ and $35x^2y$
5. $52r^2s$ and $78rs^2t$	6. $98a^2b^2c$ and $-70abc^2$
7. $22xy^2z^2$, $33x^2yz^2$, and $44x^2yz$	8. $200a^3b^2c$, $300a^2bc^3$, and $400ab^3c^2$

Factoring Quadratic Polynomials

In this section we will begin reviewing and learning an assortment of methods for factoring polynomials.

Method #1: Factoring using the GCF

Procedure for factoring a monomial out of a polynomial using the Greatest Common Factor:

Step 1: Find the GFC of each term in the polynomial.

Step 2: Divided each term by that GCF.

Step 3: Write the polynomial as the product of the GCF and the answer to the division problem

Examples:

1. $3x^2 + 12x$ GCF:
2. $7y^2 + 14y + 28$ GCF:
3. $21a^3 - 14a^2$ GCF:
4. $3a^3 + 6a^2 - 12a$ GCF:
5. $-10x^2y - 15xy^2$ GCF:
6. $20y^4 + 35y^3 + 15y^2$ GCF:

Method #2: Factoring quadratic polynomials, when $a = 1$.

Consider: $x^2 + 6x + 5$

Procedure for factoring quadratic polynomials, when $a = 1$.

Step 1:

Step 2:

Examples:

1. $x^2 - 7x + 6$
2. $x^2 + 9x + 14$
3. $x^2 + 2x - 3$
4. $x^2 - 4x - 21$

Prime Polynomials: Polynomials that cannot be factored.

Examples: $p^2 - 8p + 9$ $\quad$ $p^2 + 6p - 8$

Break for Practice: Factor completely. Hint – always look for common factors first.

1. $x^2 + 5x + 6$
2. $x^2 - 9x + 14$
3. $x^2 - x - 12$
4. $x^2 + 7x - 12$
5. $2x^2 + 4x - 48$
6. $3x^2 - 27x + 24$

Extended Practice: Factor Completely

1. $x^2 - 9x + 8$	2. $x^2 + 9x + 14$
3. $x^2 - 11x + 18$	4. $x^2 - 10x + 9$

5. $x^2 + 12x + 20$	6. $3x^2 - 15x + 18$
7. $p^2 - 8p + 9$	8. $2h^2 - 20h + 48$
9. $s^2 - 20s + 36$	10. $z^2 - 9z + 12$
11. $5x^2 + 5x - 60$	12. $t^2 + 2t - 15$
13. $x^2 - 2x - 35$	14. $s^2 - 6s - 27$

Method #3: Factoring quadratic polynomials, when $a \neq 1$ (Grouping- Splitting the Middle)

Procedure for factoring quadratic polynomials, when $a \neq 1$

Step 1: Multiply the ___________________ of the quadratic term and the ___________________ together

Step 2: Find the set of ___________________ from the number found in ___________________ that add to the ___________________ term

Step 3: ___________________ with the original ___________________ term and ___________________. Replace the ___________________ the two factors found in step #2

Step 4: ___________________ the first two terms together- Factor out the GCF.
___________________ the second two terms together- Factor out the GCF.

Step 5: Rewrite as two binomials

Example: Factor $9x^2 - 56x + 12$.

Break for Practice: Factor by splitting the middle.

1. $3x^2 - 16x + 13$
2. $18x^2 + 27x + 10$
3. $6x^2 + 7x - 10$
4. $10x^2 + 7xy + y^2$

Extended Practice: Factor by splitting the middle

1. $8x^2 - 79x - 10$	2. $12x^2 + 25x + 12$	3. $18x^2 - 15x + 2$
4. $4x^2 - 12x + 9$	5. $15x^2 + 8x - 12$	6. $25x^2 + 20xy + 4y^2$

Method #4: Difference of Squares

Consider the following polynomial. $x^2 - 25$. Is this equivalent to $x^2 + 0x - 25$? Factor it.

Result: **Difference of Two Squares** → $a^2 - b^2 = (a + b)(a - b)$

Break for Practice: Factor Completely.

1. $49x^2 - 4$
2. $27 - 3h^2$

Extended Practice: Factor Completely

1. $16k^2 - 1$	2. $3z^2 + 4z + 1$
3. $5v^2 + 4v - 1$	4. $x^2 - xy - 30y^2$
5. $p^2 + 2pq - 24q^2$	6. $16x^2 - 25$

7. $81 - 4h^2$	8. $u^2 - 8uv - 12v^2$
9. $4r^2 + 8r + 3$	10. $6s^2 + st - 5t^2$
11. $2h^2 + 7hk - 15k^2$	12. $36p^2 - 49q^2$

Method #5: Sums and Differences of Cubes

We saw how to factor the difference of two perfect squares, but what happens if we are working with perfect cubes instead?

To understand how perfect cubes work, we will first look at a multiplication problem. The result will give us clues for factoring perfect cubes.

Multiply: $(x + 5)(x^2 - 5x + 25)$

Result: $\boldsymbol{a^3 + b^3} =$ ____ similarly

$\boldsymbol{a^3 - b^3} =$ ____

Break for Practice: Factor Completely

1. $x^3 + 64$
2. $2n^3 - 54$
3. $2x^3 + 250$

Extended Practice: Factor Completely. (These are a mixture of the different types we have studied.)

1. $x^3 - 8$	2. $x^2 - 16$
3. $x^2 + 5x - 14$	4. $64x^3 + 125$
5. $3x^2 + 9x - 30$	6. $x^2 - 9x + 20$
7. $25x^2 - 9$	8. $2x^2 - 4x - 70$
9. $8x^3 - 27$	10. $4x^2 - 12x + 9$

Solving Polynomial Equations

In this section we will use the factoring techniques you have learned to solve polynomial equations. A key property to remember when solving these is if $ab = 0$, then $a = 0$, or $b = 0$, or both = 0.

Steps for Solving Polynomial Equations:

1. Set the equation equal to _______________.
2. ______________________ the equation.
3. Set each _____________________ equal to _________________ and solve.

Break for Practice: Solve each equation. The answers are called "roots", "zeros", or "solutions".

1. $x^2 + 7x = 18$

2. $3r^2 = 10r + 8$

3. $x^2 + 25 = 10x$ The solution(s) to this is called a double root. Why?

4. $(a + 3)(a - 3) = 40$

5. $(c - 6)^2 = c$

Extended Practice: Solve each equation.

1. $(x-1)(x-4)=0$

2. $t(t+1)(t-2)=0$

3. $z^2+3=4z$

4. $x^2-12=4x$

5. $3r^2=4r-1$

6. $6x^2=1-x$

7. $6(x+12)=x^2$

8. $(u+3)(u-3)=8u$

9. $3t(t+1) = 4(t+1)$

10. $2(r^2+1) = 5r$

Problem Solving Using Polynomial Equations

Now it's time to see where polynomial equations can be used.

Example: A rectangle is 5 m longer than it is wide, and its area is 176 m^2. Find its dimensions.

Example: The hypotenuse of a right triangle is 13 in long. One leg is 7 in longer than the other leg. Find the length of each leg.

Vertical motion (ex: a thrown ball, a rocket, etc.) affected only by gravity leads to a formula that is a polynomial.

h = height   v = initial velocity   t = time in seconds

If h is in meters, then $h = vt - 4.9t^2$.   If h is in feet, then $h = vt - 16t^2$.

Example: A batter hits a baseball directly upward with a speed of 96 ft/sec.

a) How long is the ball in the air before being caught by the catcher?

b) How high did the ball go?

Extended Practice: Solve

1. A rectangle is 4 cm longer than it is wide, and its area is 117 cm^2. Find its dimensions.

2. The top of a 15-foot ladder is 3 ft. farther up a wall than the foot of the ladder is from the bottom of the wall. How far is the foot of the ladder from the bottom of the wall?

3. A rectangle is 15 cm wide and 18 cm long. If both dimensions are decreased by the same amount, the area of the new rectangle formed is 116 cm^2 less than the area of the original. Find the dimensions of the new rectangle.

4. A projectile is launched upward from ground level with an initial speed of 98 m/sec.

   a) When will it return to the ground?

b) How high will it go?

5. A ball is thrown directly upward from ground level with an initial speed of 80 ft/sec.

a) When will it return to the ground?

b) How high will it go?

Solving Polynomial Inequalities

The final section in this unit is solving polynomial inequalities. The use of factoring and graphing are the keys to solving these problems.

What does $ab > 0$ *and* $ab < 0$ mean?

$\boldsymbol{ab > 0}$ → means…

** So,

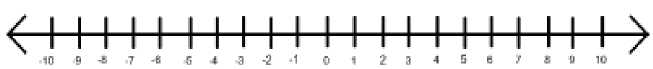
$\boldsymbol{ab < 0}$ → means…

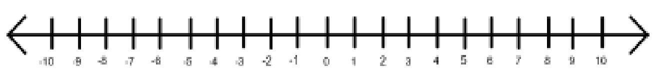
** So,

Example: Find and graph the solution set of $x^2 + 3x < -2$.

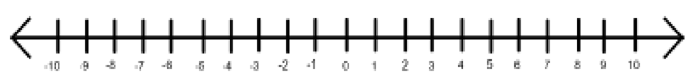
Steps:

1. Set one side of the inequality to zero.
2. Factor
3. Identify the "breaking points"
4. Graph separate number lines and combine



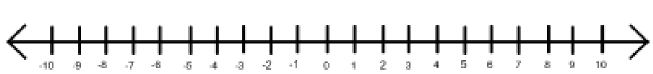


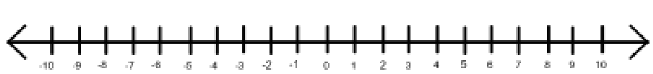
Solution set: ________________________________



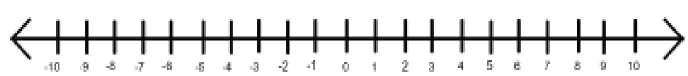
Break for Practice: Find and graph the solution set to each inequality.

1. $x^2 + 15x \leq 4x - 24$

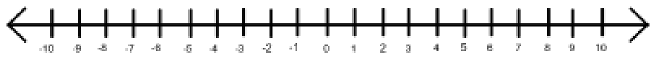




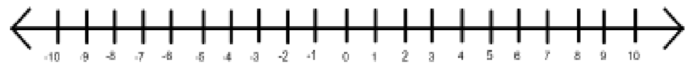
Solution set: ________________________________



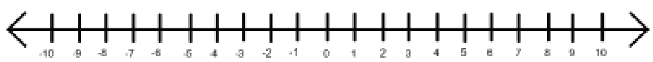
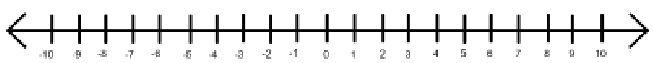
2. $9r(r-1) \geq -2$



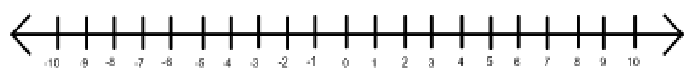

Solution set: ___________________________________



3. $25x - x^3 > 0$

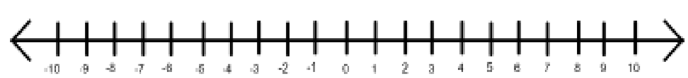
Solution set: ___________________________________



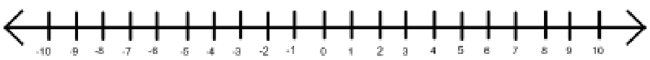
Extended Practice: Find and graph the solution set to each inequality.

1. $4x(x+1) \geq 3$

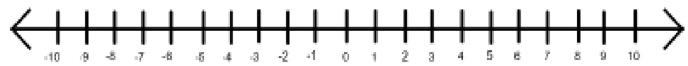

Solution set: ___________________________________



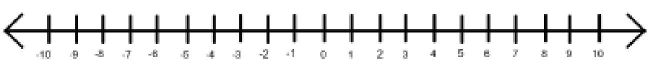
2. $12 + s - s^2 \geq 0$

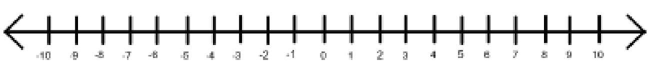


Solution set: ______________________________

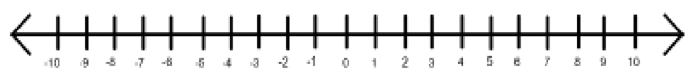


3. $4y^2 < 36$

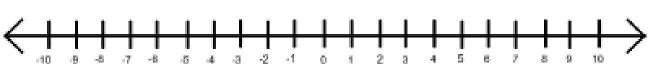


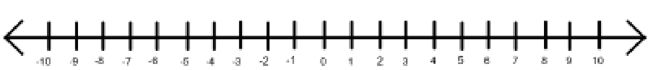


Solution set: ______________________________

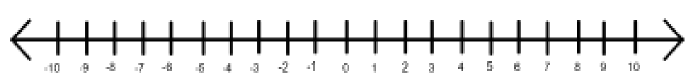


4. $r^2 \leq 2(r + 4)$

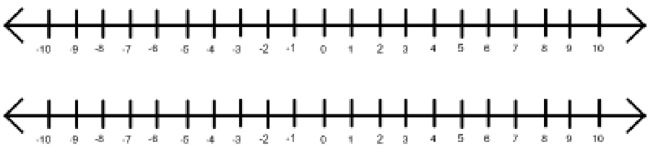




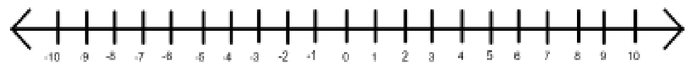
Solution set: ______________________________



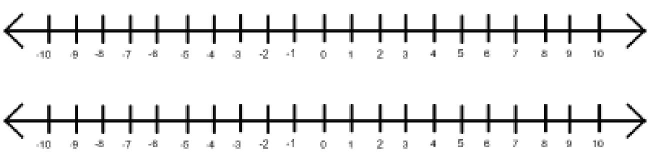
5. $y^3 + y^2 < 6y$



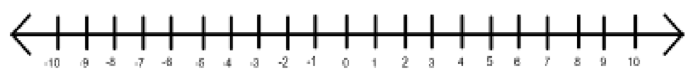
Solution set: ___________________________________



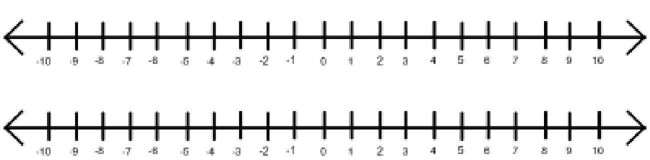
6. $z^3 + 7z^2 + 10z > 0$



Solution set: ___________________________________



7. $4z(z-1) \leq 15$



Solution set: ___________________________________