Solving Systems with Gaussian Elimination 3x3

There is one more method for solving systems that we will learn. This is called Gaussian Elimination and it is named for the German mathematician Carl Friederich Gauss. The first example we will try is the same first example that we had for linear combination. If we get the same answer, then we will know that this method truly works.

Break for Practice: Solve the system using Gaussian Elimination.

1. $2x - y - z = 7$
   $3x + 5y + z = -10$
   $4x - 3y + 2z = 4$ $\rightarrow$

$$\begin{array}{cccc} x & y & z & c \end{array}$$
$$\left[\begin{array}{ccc|c} 2 & -1 & -1 & 7 \\ 3 & 5 & 1 & -10 \\ 4 & -3 & 2 & 4 \end{array}\right]$$

Goal #1: Get a 1 in top left position

$Row_2 - Row_1 \rightarrow Row_1$

$$\begin{array}{cccc} -3 & -18 & -6 & 51 \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 6 & 2 & -17 \\ 3 & 5 & 1 & -10 \\ 4 & -3 & 2 & 4 \end{array}\right]$$

Goal #2: Get zeros below the one (from Step #1)

$-3(Row_1) + Row_2 \rightarrow Row_2$

$$\begin{array}{cccc} -4 & -24 & -8 & 68 \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 6 & 2 & -17 \\ 0 & -13 & -5 & 41 \\ 4 & -3 & 2 & 4 \end{array}\right]$$

$-4(Row_1) + Row_3 \rightarrow Row_3$

$$\left[\begin{array}{ccc|c} 1 & 6 & 2 & -17 \\ \overset{0}{0} & \overset{78}{-13} & \overset{30}{-5} & \overset{-246}{41} \\ \overset{0}{0} & \overset{-135}{-27} & \overset{30}{-6} & \overset{+360}{72} \end{array}\right]$$

Goal #3: Use both bottom rows to get 1 more zero.

$-6(Row_2) + 5(Row_3) \rightarrow Row_3$

$$\begin{array}{cccc} x & y & z & c \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 6 & 2 & -17 \\ 0 & -13 & -5 & 41 \\ 0 & -57 & 0 & 114 \end{array}\right]$$

$$-57y = 114$$
$$\frac{-57y}{-57} = \frac{114}{-57}$$
$$y = -2$$

$$-13(-2) - 5z = 41$$
$$26 - 5z = 41$$
$$-26 \qquad -26$$
$$-5z = 15$$
$$\frac{-5z}{-5} = \frac{15}{-5}$$
$$z = -3$$

$$x + 6(-2) + 2(-3) = -17$$
$$x - 12 - 6 = -17$$
$$x - 18 = -17$$
$$+18 \qquad +18$$
$$x = 1$$

$$\boxed{\text{Solution: } (\underset{x}{1}, \underset{y}{-2}, \underset{z}{-3})}$$

☆ Do 1st in Class ☆

2. $x - y + 3z = -4$
$-x + 2y + z = 2$
$3x - 4y - z = -4$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 3 & -4 \\ -1 & 2 & 1 & 2 \\ 3 & -4 & -1 & -4 \end{array}\right] \leftarrow$$

Goal #1: Get a 1 in top left position (done)

Row₁ + Row₂ → Row₂

$$\begin{array}{cccc} -3 & 3 & -9 & 12 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & -4 \\ 0 & 1 & 4 & -2 \\ 3 & -4 & -1 & -4 \end{array}\right]$$

← Goal #2: Get zeros below the one (from Step #1)

−3(Row₁) + Row₃ → Row₃

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & -4 \\ 0 & 1 & 4 & -2 \\ 0 & -1 & -10 & 8 \end{array}\right]$$

Row₂ + Row₃ → Row₃

$$\begin{array}{c} \begin{array}{cccc} x & y & z & = c \end{array} \\ \left[\begin{array}{ccc|c} 1 & -1 & 3 & -4 \\ 0 & 1 & 4 & -2 \\ 0 & 0 & -6 & 6 \end{array}\right] \end{array}$$

← Goal #3: use both bottom rows to get 1 more zero.

$-6z = 6$
$\frac{\quad}{-6} \quad \frac{\quad}{-6}$

$\underbrace{z = -1}$

$y + 4(-1) = -2$

$y - 4 = -2$
$+4 \quad +4$

$\underbrace{y = 2}$

$x - (2) + 3(-1) = -4$

$x - 2 - 3 = -4$

$x - 5 = -4$
$+5 \quad +5$

$\underbrace{x = 1}$

$$\boxed{\text{Solution: } (\underset{x}{1}, \underset{y}{2}, \underset{z}{-1})}$$

Extended Practice: Solve each system using Gaussian Elimination.

1. $x + 2y - 3z = 11$
   $2x + y - 2z = 9$
   $4x + 3y + z = 16$

   $(2, 3, -1)$

2. $2x - y + 3z = 19$
   $x + 3y - z = -10$
   $3x + 5y + 2z = 3$

   $(1, -2, 5)$

3. $2x + 3y - z = 9$
$x - 3y + z = -6$
$3x + y - 4z = 31$

$(1, 0, -7)$

4. $4x + 3y - 5z = -26$
$2x + y - z = -4$
$x + 5y + z = -38$

$(4, -9, 3)$

Problem Solving Using Linear Systems

In this section we will see how systems can be used to solve applications. Some of these problems are similar to problems that we had seen earlier and solved using one variable, now we can learn how to solve them with two variables.

★ Use Substitution or Elimination Methods ★

Break for Practice: Solve

1. Tickets for the senior prom cost $25 for a single ticket and $40 for a couple. Ticket sales totaled $3800 and 110 tickets were sold. How many tickets of each type were sold?

total #1 total #2

x = # of single tickets

y = # of couple tickets

Equation #1: $\$25x + \$40y = \$3800$

Equation #2: $x + y = 110 \rightarrow y = 110 - x$

Elimination (x -25):

$25x + 40y = 3800$
$-25x - 25y = -2750$
$15y = 1050$
$\frac{15y}{15} = \frac{1050}{15}$
$y = 70$

$x + 70 = 110$
$-70 \quad -70$
$x = 40$

Substitution:

$25x + 40(110 - x) = 3800$
$25x + 4400 - 40x = 3800$
$-4400 \quad -4400$
$-15x = -600$
$\frac{-15x}{-15} = \frac{-600}{-15}$
$x = 40$

$y = 110 - 40$
$y = 70$

$\boxed{\text{Solution: 40 single tickets, 70 couple tickets}}$

2. A cashier had to give Sarah $3.45 in change but he had only quarters and dimes in the cash register. If he gave her 15 coins, how many dimes did she receive?

total #1 total #2

d = # of dimes

q = # of quarters

Equation #1: $0.10d + 0.25q = 3.45$

Equation #2: $d + q = 15 \rightarrow d = 15 - q$

Elimination (x -3.75):

$0.10d + 0.25q = 3.45$
$0.25d - 0.25q = -3.75$
$-0.15d = -0.30$
$\frac{-0.15d}{-0.15} = \frac{-0.30}{-0.15}$
$d = 2$

$2 + q = 15$
$-2 \quad -2$
$q = 13$

Substitution:

$0.10(15 - q) + 0.25q = 3.45$
$1.5 - 0.10q + 0.25q = 3.45$
$1.5 + 0.15q = 3.45$
$-1.50 \quad -1.50$
$0.15q = 1.95$
$\frac{0.15q}{0.15} = \frac{1.95}{0.15}$
$q = 13$

$d = 15 - 13$
$d = 2$

$\boxed{\text{Solution: 2 dimes}}$

3. A grocer mixed nuts worth $4 per kilogram with raisins worth $3.25 per kilogram to make 15 kilograms of a mixture worth $3.60 per kilogram. How many kilograms of nuts were used?

total #1

total cost $= 15(3.60) = \$54$

n = # Kg of nuts

r = # Kg of raisins

Equation #1: $n + r = 15 \rightarrow r = 15 - n$

Equation #2: $\$4n + \$3.25r = \$54$

Elimination (x -3.25):

$-3.25n - 3.25r = -48.75$
$4n + 3.25r = 54$
$0.75n = 5.25$
$\frac{0.75n}{0.75} = \frac{5.25}{0.75}$
$n = 7$

$7 + r = 15$
$-7 \quad -7$
$r = 8$

Substitution:

$4n + 3.25(15 - n) = 54$
$4n + 48.75 - 3.25n = 54$
$-48.75 \quad -48.75$
$0.75n = 5.25$
$\frac{0.75n}{0.75} = \frac{5.25}{0.75}$
$n = 7$

$r = 15 - 7$
$r = 8$

$\boxed{\text{Solution: 7 Kg of nuts}}$

Extended Practice: Solve

1. Kerry asked a bank teller to cash a $390 check using $20 bills and $50 bills. If the teller gave her a total of 15 bills, how many of each type of bill did she receive? 12 - $20 Bills 3 - $50 Bills
2. Tickets for the homecoming dance cost$20 for a single ticket or $35 for a couple. Ticket sales totaled $2280, and 128 people attended. How many tickets of each type were sold? 16 - Single Tickets 56 - Couple Tickets
3. On Friday, the With-It Clothiers sold some jeans at $25 a pair and some shirts at $18 each. Receipts for the day totaled $441. On Saturday the store priced both items at $20, sold exactly the same number of each item, and had receipts of $420. How many pairs of jeans, and how many shirts were sold each day? 9 - Jeans 12 - Shirts

4. A grain-storage warehouse has a total of 30 bins. Some hold 20 tons of grain each, and the rest hold 15 tons each. How many of each type of bin are there if the capacity of the warehouse is 510 tons?

18 - Small bins

12 - large bins

Break for Practice: Solve

wind is against you ($a-w$) ★ Rate x Time = Distance ★ wind is with you ($a+w$)

1. With a head wind, a plane travelled 840 miles northward in 2 hours. With the same wind as a tail wind, the return trip southward took 1 hour 45 minutes. Find the plane's air speed and the wind speed.

a = air speed

w = wind speed

★ 1 hour and 45 minutes = 1.75 hours

Equation #1 (against wind): $\frac{2(a-w)}{2} = \frac{840}{2} \xrightarrow{\div 2} a-w = 420$

Equation #2 (with wind): $\frac{1.75(a+w)}{1.75} = \frac{840}{1.75} \xrightarrow{\div 1.75} a+w = 480$

$\frac{2a}{2} = \frac{900}{2}$

$a = 450$

$450 + w = 480$
$-450 \quad\quad -450$

$w = 30$

$\boxed{\text{Solution: air speed is 450 mph, wind speed is 30 mph}}$

2. A garage charges a fixed amount to bring your car to be fixed, and an additional hourly fee for the mechanic to work on your car. Keith brings his car in and received a bill for $160 (total #1) after the mechanic worked on his car for 4 hours. Mildred had to pay $435 (total #2) after her car was worked on for 15 hours. What are the hourly and fixed rates?

h = hourly rate

f = fixed rate

Equation #1 (Keith): $f + 4h = \$160$ → $f = 160 - 4h$

Equation #2 (Mildred): $f + 15h = \$435$

Elimination (mult. -1):

$-f - 4h = -160$
$f + 15h = 435$

$\frac{11h}{11} = \frac{275}{11}$

$h = 25$

$f + 4(25) = 160$

$f + 100 = 160$
$-100 \quad -100$

$f = 60$

$\boxed{\text{Solution: Fixed - \$60, Hourly - \$25}}$

Substitution:

$160 - 4h + 15h = 435$
$-160 \quad\quad -160$

$\frac{11h}{11} = \frac{275}{11}$

$h = 25$

$f = 160 - 4(25)$

$f = 160 - 100$

$f = 60$

Extended Practice: Solve

1. With a tail wind, a helicopter traveled 300 miles in 1 hour 40 minutes. The return trip against the same wind took 20 minutes longer. Find the wind speed and also the air speed of the helicopter.

Air Speed = 165 mph

Wind Speed = 15mph

2. With a head wind, a plane traveled 1000 miles in 4 hours. With the same wind as a tail wind, the return trip took 3 hours 20 minutes. Find the plane's air speed and the wind speed.

Air Speed = 275 mph

Wind Speed = 25mph

3. A caterer's total cost for catering a party includes a fixed cost, which is the same for every party. In addition, the caterer charges a certain amount for each guest. If it costs \$300 to serve 25 guests and \$420 to serve 40 guests, find the fixed cost and the cost per guest.

Fixed Rate: \$ 100

Cost per guest: \$8