8. $t_2 = 18$ $t_3 = 12$ $t_5 =$ $5\frac{1}{3}$	9. $t_3 = -12$ $t_6 = 96$ $t_9 =$ -768
$\frac{12 = t_1 \cdot r^{3-1}}{18 = t_1 \cdot r^{2-1}} \rightarrow \frac{12 = t_1 \cdot r^2}{18 = t_1 \cdot r}$ $\frac{2}{3} = r$ $\left(\frac{3}{2}\right) 18 = t_1 \left(\frac{2}{3}\right)$ $27 = t_1$ $t_5 = 27 \cdot \left(\frac{2}{3}\right)^{5-1}$ $t_5 = 5\frac{1}{3}$	$\frac{96 = t_1 \cdot r^{6-1}}{-12 = t_1 \cdot r^{3-1}} \rightarrow \frac{96 = t_1 \cdot r^5}{-12 = t_1 \cdot r^2}$ $\sqrt[3]{-8} = \sqrt[3]{r^3}$ $-2 = r$ $-12 = t_1 \cdot (-2)^2$ $\frac{-12}{4} = \frac{4t_1}{4}$ $t_1 = -3$ $t_9 = -3 \cdot (-2)^{9-1}$ $t_9 = -768$

Now let's look at a couple of other types of problems that we can solve with the geometric sequence formula.

Break for Practice:

1. Find the position, n, of the underlined term in each geometric sequence.

a.) $\frac{1}{9}, \frac{1}{3}, 1, 3, \ldots, \underline{19683}, \ldots$

×3

$t_1 = \frac{1}{9}$

$r = 3$

$t_n = 19683$

$\frac{19683}{1/9} = \frac{\frac{1}{9} \cdot 3^{n-1}}{1/9}$

$177147 = 3^{n-1}$

$\log_3 177147 = n-1$

$n-1 = \frac{\log 177147}{\log 3}$

$n-1 = 11$
$+1 \quad +1$

$n = 12$

b.) $17, 34, 68, 136, \ldots, \underline{34{,}816}, \ldots$

×2

$t_1 = 17$

$r = 2$

$t_n = 34{,}816$

$\frac{34{,}816}{17} = \frac{17 \cdot 2^{n-1}}{17}$

$2048 = 2^{n-1}$

$\log_2 2048 = n-1$

$n-1 = \frac{\log 2048}{\log 2}$

$n-1 = 11$
$+1 \quad +1$

$n = 12$

Geometric Means – terms between two given terms in a geometric sequence.

2. Find the stated number of geometric means between the two given terms.

a.) Four between 4 and 972.

4, _, _, _, _, 972
$t_1 \qquad t_6$

$t_1 = 4$

$t_6 = 972$

$n = 6$

$\frac{972}{4} = \frac{4 \cdot r^{6-1}}{4}$

$\sqrt[5]{243} = \sqrt[5]{r^5}$

$r = 3$

$\boxed{4, \underline{12}, \underline{36}, \underline{108}, \underline{324}, 972}$

b.) Three between 3 and 48.

3, _, _, _, 48
$t_1 \qquad t_5$

$t_1 = 3$

$t_5 = 48$

$n = 5$

$\frac{48}{3} = \frac{3 \cdot r^{5-1}}{3}$

$\sqrt[4]{16} = \sqrt[4]{r^4}$

$\pm 2 = r$

3, 6, 12, 24, 48

3, −6, 12, −24, 48

Extended Practice: Find the position, n, of the underlined term in each geometric sequence.

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1. 5, -25, 125, -625, …, <u>3125</u> ,…

$t_1 = 5$

$r = -5$

$t_n = 3125$

$\frac{3125}{5} = \frac{5 \cdot (-5)^{n-1}}{5}$

$625 = -5^{n-1}$

$\log_{-5} 625 = n-1$

$n-1 = \frac{\log 625}{\log -5}$ ← use 5 in calc. (neg just alt. the signs in the sequence)

$n - 1 = 4$

$+1 \quad +1$

$n = 5$

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2. 27, 9, 3, …, $\underline{\frac{1}{81}}$, …

$\div 3$

$t_1 = 27$

$r = 1/3$

$t_n = \frac{1}{81}$

$\frac{\frac{1}{81}}{27} = \frac{27 \cdot 1/3^{n-1}}{27}$

$\frac{1}{81} \cdot \frac{1}{27} = 1/3^{n-1}$

$\log_{1/3} \frac{1}{2187} = n-1$

$n-1 = \frac{\log \frac{1}{2187}}{\log 1/3}$ → $n-1 = 7$

$n = 8$

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Find the stated number of geometric means between the two given terms.

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3. One between 2 and 8

2, —, 8

$t_1 \quad t_3$

$t_1 = 2$

$t_3 = 8$

$n = 3$

$\frac{8}{2} = \frac{2 \cdot r^{3-1}}{2}$

$\sqrt[2]{4} = \sqrt[2]{r^2}$

$\pm 2 = r$

$\boxed{2, 4, 8 \text{ OR } 2, -4, 8}$

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4. One between -18 and -36

-18, —, -36

$t_1 \quad t_3$

$t_1 = -18$

$t_3 = -36$

$n = 3$

$\frac{-36}{-18} = \frac{-18 \cdot r^{3-1}}{-18}$

$\sqrt{2} = \sqrt{r^2}$

$r = \pm\sqrt{2}$

$\boxed{-18, -18\sqrt{2}, -36 \text{ OR } -18, 18\sqrt{2}, -36}$

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5. Three between 5 and 80

5, —, —, —, 80

$t_1 \quad t_5$

$t_1 = 5$

$t_5 = 80$

$n = 5$

$\frac{80}{5} = \frac{5 \cdot r^{5-1}}{5}$

$\sqrt[4]{16} = \sqrt[4]{r^4}$

$\pm 2 = r$

$\boxed{5, 10, 20, 40, 80 \text{ OR } 5, -10, 20, -40, 80}$

</td>
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6. Two between -4 and 108

-4, —, —, 108

$t_1 \quad t_4$

$t_1 = -4$

$t_4 = 108$

$n = 4$

$\frac{108}{-4} = \frac{-4 \cdot r^{4-1}}{-4}$

$\sqrt[3]{-27} = \sqrt[3]{r^3}$

$-3 = r$

$\boxed{-4, 12, -36, 108}$

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Tell whether each sequence is arithmetic or geometric. Then find a formula for the sequence.

7. The sequence of positive even integers. 2, 4, 6, 8, 10 . . . Arithmetic $\Rightarrow$ $t_n = 2 + (n-1)2$ $t_1 = 2$ $t_n = 2 + 2n - 2$ $d = 2$ $t_n = 2n$	8. 25, 33, 41, 49, … +8 Arithmetic $\Rightarrow$ $t_n = 25 + (n-1)8$ $t_1 = 25$ $t_n = 25 + 8n - 8$ $d = 8$ $t_n = 8n + 17$
9. 200, -100, 50, -25, … $\div -2$ Geometric $\Rightarrow$ $t_n = 200 \cdot (-\frac{1}{2})^{n-1}$ $t_1 = 200$ $r = -\frac{1}{2}$	

Story Problems with Sequences

In this section we shall see arithmetic and geometric sequences applied to various story problems. You will need to decide if the situation described is arithmetic or geometric in order to solve it.

Break for Practice:

Adding \$250 each year $\rightarrow$ Arithmetic

1. A part time teacher takes a position at \$6,600 per year. He receives annual increases of \$250. What will his salary be during his fifteenth year of service?

$t_1 = 6,600$ $t_{15} = 6,600 + (15-1)250$

$d = 250$

$n = 15$ $t_{15} = \$10,100$

$\leftarrow$ term 1

2. A wealthy man gave his son \$5 on his tenth birthday and decided to double his gift each following year. How much did the boy receive on his 21st birthday?

multiplying by 2 $\Rightarrow$ Geometric

$t_1 = 5$ $t_{12} = 5 \cdot 2^{12-1}$

$r = 2$ $t_{12} = \$10,240$

$n = 12 \leftarrow 21 - 9$

term #:	1	2	3	difference of 9
age	10	11	12	

3. A new house purchased for $125,000 is expected to increase in value by 3 % per year. What should its value be in 12 years?

Multiplied by 1.03

$t_1 = \$125{,}000$ $\quad t_{12} = 125{,}000 \cdot 1.03^{13-1}$

$r = 1.03$ $\quad t_{12} = \$178220.11$

$n = 13$

this year = term 1 → in 12 years = 13th year

+0.03

0.35, 0.38

4. A well drilling firm charges $0.35 to drill the first foot, $0.38 for the second foot, and so on in an arithmetic progression. At this rate, how much does the firm charge to drill the last foot of a well 350 feet deep?

$t_1 = 0.35$ $\quad t_{350} = 0.35 + (350-1)\,0.03$

$d = 0.03$

$n = 350$ $\quad t_{350} = \$10.82$

Extended Practice:

1. Allysa has taken a job with a starting salary of $17,600 and annual raises of $850. What will be her salary during her fifth year on the job?

$t_1 = \$17{,}600$ $\quad t_5 = 17{,}600 + (5-1)850$

$d = \$850$ $\quad t_5 = \$21{,}000$

$n = 5$

2. Frank has taken a job with a starting salary of $15,000 and annual raises of 4%. What will be his salary during his third year on the job?

$t_1 = 15{,}000$ $\quad t_3 = 15{,}000 \cdot 1.04^{3-1}$

$r = 1.04$ $\quad t_3 = \$16{,}224$

$n = 3$

3. An advertisement for a mutual fund claims that people who invested in the fund 5 years ago have doubled ×2 their money. If the fund's future performance is similar to its past performance, how much would a $2,000 investment be worth in 40 years?

$t_1 = \$2{,}000$

$r = 2$

$\boxed{n = 9}$

$t_9 = 2{,}000 \cdot 2^{9-1}$

$t_9 = \$512{,}000$

term #:	1 →	2 →	3 →	4 →	5 →	6 →	7 →	8 →	9
years:	0	5	10	15	20	25	30	35	40 yrs

4. A culture of yeast doubles ×2 in size every 4 hours. If the yeast population is estimated to be 3 million now, what will it be one day from now?

$t_1 = 3$

$r = 2$

$\boxed{n = 7}$

$t_7 = 3 \cdot 2^{7-1}$

$t_7 = 192$ million

term #:	1 →	2 →	3 →	4 →	5 →	6 →	7
hours:	0	4	8	12	16	20	24 hrs

Series and Sigma (Summation) Notation

Now that we have spent several days exploring sequences, we are ready to explore the topic of series. First we need to understand what a series is.

Example: Sequence : 1, 3, 5, 7, 9, …

Related Series: 1+ 3 + 5 + 7 + 9 + …

Definition: A **Series** is the sum of the terms in a sequence

Now, since series have an infinite number of terms, most of them will have an infinite sum. Because of this, it is usually more interesting to consider partial sums.

Definition: A **Partial Sum** is the sum of the first finite number of terms in a series.

<u>Notation</u>: S_n stands for the n^{th} partial sum, which is the sum of the first n terms in a series.

Example: Consider 2 + 7 + 12 + 17 + …

$S_1 =$ 2

$S_2 =$ 2 + 7 = 9

$S_3 =$ 2 + 7 + 12 = 21

Since it can take a lot of space and time to write out all of the terms, a shorthand notation was developed. This is called sigma or summation notation.

Sigma or Summation Notation:

Upper Bound

$$S_n = \sum_{k=1}^{n} t_k$$

Summand

Sigma Sign

Lower Bound

Break for Practice:

1. Expand $S_5 = \sum_{k=1}^{5}(5k+3)$ = 5(1)+3 + 5(2)+3 + 5(3)+3 + 5(4)+3 + 5(5)+3

= 8 + 13 + 18 + 23 + 28

Sum = 90

2. Expand $S_4 = \sum_{k=1}^{4} 128\left(\frac{1}{2}\right)^{k-1}$ = $128\left(\frac{1}{2}\right)^{1-1} + 128\left(\frac{1}{2}\right)^{2-1} + 128\left(\frac{1}{2}\right)^{3-1} + 128\left(\frac{1}{2}\right)^{4-1}$

= 128 + 64 + 32 + 16

Sum = 240

Extended Practice: Expand each of the following partial sums.

1. $\sum_{k=1}^{6}(k+10)$ = 11 + 12 + 13 + 14 + 15 + 16 Sum = 81

2. $\sum_{k=1}^{8} 3k$ = 3 + 6 + 9 + 12 + 15 + 18 + 21 + 24 Sum = 108

3. $\sum_{k=1}^{6} 2^k = 2 + 4 + 8 + 16 + 32 + 64$

Sum: 126

4. $\sum_{k=4}^{10} (3k-2) = 3(4)-2 + 3(5)-2 + 3(6)-2 + 3(7)-2 + 3(8)-2 + 3(9)-2 + 3(10)-2$

$= 10 + 13 + 16 + 19 + 22 + 25 + 28$

Sum: 133

5. $\sum_{k=0}^{5} \frac{(-1)^k}{k+1} = \frac{(-1)^0}{0+1} + \frac{(-1)^1}{1+1} + \frac{(-1)^2}{2+1} + \frac{(-1)^3}{3+1} + \frac{(-1)^4}{4+1} + \frac{(-1)^5}{5+1}$

$= \frac{1}{1} + \left(\frac{-1}{2}\right) + \frac{1}{3} + \left(\frac{-1}{4}\right) + \frac{1}{5} + \left(\frac{-1}{6}\right)$

Sum: $\frac{37}{60} = 0.61\overline{6}$

6. $\sum_{k=0}^{3} 4^{-k} = 4^{-0} + 4^{-1} + 4^{-2} + 4^{-3}$

$= 1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64}$

Sum: $\frac{85}{64} \approx 1.328125$

7. $\sum_{k=3}^{8} |5-k| = |5-3| + |5-4| + |5-5| + |5-6| + |5-7| + |5-8|$

$= 2 + 1 + 0 + 1 + 2 + 3$

Sum: 9

8. $\sum_{k=1}^{4} (-k)^{k+1} = (-1)^{1+1} + (-2)^{2+1} + (-3)^{3+1} + (-4)^{4+1}$

$= 1 + (-8) + 81 + (-1024)$

Sum: -950

Now we will try to go in the reverse direction. We will rewrite a series from expanded form into sigma notation. It will be useful on many of the problems to remember the formulas for arithmetic and geometric sequences.

Review: Arithmetic Sequence formula: $t_n = t_1 + (n-1)d$

Geometric Sequence formula: $t_n = t_1 \cdot r^{n-1}$

Break for Practice: Rewrite each series into sigma notation.

1. $3 + 10 + 17 + 24 + \ldots + 66$ ← upper bound (term value looking for n)

$+7$

Arithmetic ⇒ $t_n = t_1 + (n-1)d$

$t_n = 3 + (n-1)7$

$t_n = 3 + 7n - 7$

$t_n = \underbrace{7n - 4}_{\text{summand}}$

$66 = 7n - 4$
$+4 \qquad +4$

$\frac{70}{7} = \frac{7n}{7}$

$\underbrace{10 = n}$ ← upper bound

$$\boxed{\sum_{n=1}^{10} 7n - 4}$$

2. $3 + 12 + 48 + 192 + \ldots + 12{,}582{,}912$ ← upper bound (term value looking for n.)

$\times 4$

Geometric ⇒ $t_n = t_1 \cdot r^{n-1}$

$t_n = \underbrace{3 \cdot 4^{n-1}}_{\text{summand}}$

$\frac{12{,}582{,}912}{3} = \frac{3 \cdot 4^{n-1}}{3}$

$4194304 = 4^{n-1}$

$\log_4 4194304 = n-1$

$n - 1 = \frac{\log 4194304}{\log 4}$

$n - 1 = 11$
$+1 \qquad +1$

$\underbrace{n = 12}$ ← upper bound

$$\boxed{\sum_{n=1}^{12} 3 \cdot 4^{n-1}}$$

3. $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{17}$

$t_1 \quad t_2 \quad t_3 \quad t_4$ $\quad t_{17}$ ← term # (upper bound)

neither: notice the term# and the denominator are the same so

$$\boxed{\sum_{n=1}^{17} \frac{1}{n}}$$

4. $3 + 9 + 27 + 81 + \ldots$ ← no limit (∞)

$\times 3$

Geometric ⇒ $t_n = t_1 \cdot r^{n-1}$

$t_n = 3 \cdot 3^{n-1}$

$t_n = 3^1 \cdot 3^{n-1}$

$t_n = 3^{1+n-1}$

$t_n = 3^n$

$$\boxed{\sum_{n=1}^{\infty} 3 \cdot 3^{n-1}}$$

OR

$$\boxed{\sum_{n=1}^{\infty} 3^n}$$

5. $-5 + 10 - 20 + 40 - \ldots$ ← no limit (∞)

$\times -2$

Geometric ⇒ $t_n = t_1 \cdot r^{n-1}$

$t_n = \underbrace{-5 \cdot (-2)^{n-1}}_{\text{summand}}$

$$\boxed{\sum_{n=1}^{\infty} -5 \cdot (-2)^{n-1}}$$

6. $-6 + 10 - 14 + 18 - 22 + \ldots$ ← no limit (∞)

$+4$ but with alternating signs

Arithmetic ⇒ $t_n = t_1 + (n-1)d$

$t_n = 6 + (n-1)(4)$

$t_n = 6 + 4n - 4$

$t_n = \underbrace{4n + 2}$

$$\boxed{\sum_{n=1}^{\infty} (-1)^n (4n+2)}$$

Extended Practice: Rewrite each series into sigma notation.

1. $\underbrace{2 + 4 + 6 + \ldots + 1000}$ ← term value, looking for n (upper bound)

+2

Arithmetic ⇒ $t_n = t_1 + (n-1)d$

$t_n = 2 + (n-1)2$

$t_n = 2 + 2n - 2$

$\underbrace{t_n = 2n}_{\text{summand}}$

$1000 = 2n$

$\frac{1000}{2} = \frac{2n}{2}$

$\underbrace{500 = n}_{\text{upper bound}}$

$\boxed{\sum_{n=1}^{500} 2n}$

2. $5 + 10 + 15 + \ldots + 250$ term value, looking for n (upper bound)

+5

Arithmetic ⇒ $t_n = t_1 + (n-1)d$

$t_n = 5 + (n-1)5$

$t_n = 5 + 5n - 5$

$\underbrace{t_n = 5n}_{\text{summand}}$

$250 = 5n$

$\frac{250}{5} = \frac{5n}{5}$

$\underbrace{50 = n}_{\text{upper bound}}$

$\boxed{\sum_{n=1}^{50} 5n}$

3. $1^3 + 2^3 + 3^3 + \cdots + 20^3$

t_1 t_2 t_3 t_{20}

Neither; notice that the term # is the same as the base of exponent

$\boxed{\sum_{n=1}^{20} n^3}$

4. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{99}{100}$

t_1 t_2 t_3

Neither; notice that the term # is the same as the numerator, and the denominator can be found by the term # +1.

$\boxed{\sum_{n=1}^{99} \frac{n}{n+1}}$

5. $3 + 7 + 11 + 15 + \ldots + 399$ term value, looking for n (upper bound)

+4

Arithmetic ⇒ $t_n = t_1 + (n-1)d$

$t_n = 3 + (n-1)4$

$t_n = 3 + 4n - 4$

$\underbrace{t_n = 4n - 1}_{\text{summand}}$

$399 = 4n - 1$

$+1 \quad +1$

$400 = 4n$

$\frac{400}{4} = \frac{4n}{4}$

$\underbrace{100 = n}_{\text{upper bound}}$

$\boxed{\sum_{n=1}^{100} 4n-1}$

6. $\underbrace{1 + 2 + 4 + 8}_{\times 2} + \ldots + \underline{64}$ term value, looking for n (upper bound)

Geometric $\Rightarrow$ $t_n = t_1 \cdot r^{n-1}$

$t_n = 1 \cdot 2^{n-1}$ either one

$t_n = \underbrace{2^{n-1}}$ summand

$64 = 2^{n-1}$

$\log_2 64 = n-1$

$n - 1 = \dfrac{\log 64}{\log 2}$

$n - 1 = 6$

$ +1 \quad +1$

$\underbrace{n = 7}$ ← upper bound

$$\boxed{\sum_{n=1}^{7} 2^{n-1}}$$

7. $\underbrace{-9 + 3 - 1}_{\div -3} + \frac{1}{3} - \cdots$ no limit $\underbrace{(\infty)}_{\text{upper bound}}$

Geometric $\Rightarrow$ $t_n = t_1 \cdot r^{n-1}$

$t_n = \underbrace{-9 \cdot \left(-\frac{1}{3}\right)^{n-1}}_{\text{summand}}$

$$\boxed{\sum_{n=1}^{\infty} -9 \cdot \left(-\frac{1}{3}\right)^{n-1}}$$

8. $\underbrace{8 - 4 + 2}_{\div -2} - 1 + \ldots$ no limit $\underbrace{(\infty)}_{\text{upper bound}}$

Geometric $\Rightarrow$ $t_n = t_1 \cdot r^{n-1}$

$t_n = \underbrace{8\left(-\frac{1}{2}\right)^{n-1}}_{\text{summand}}$

$$\boxed{\sum_{n=1}^{\infty} 8\left(-\frac{1}{2}\right)^{n-1}}$$

8. $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$ no limit $\underbrace{(\infty)}_{\text{upper bound}}$

$t_1 \quad t_2 \quad t_3 \quad t_4$

Neither; notice that the denominator is the term # squared

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2}}$$