Name: ______________________________

Algebra II

Unit 10

Sequences and Series

<u>Priority Standard:</u>

Unit "I can" statements:

1. I can determine if a sequence is arithmetic, geometric, or neither, and I can extend a sequence.
2. I can find the formula for an arithmetic sequence, find a specified term within an arithmetic sequence, and find a given number of arithmetic means between two terms in an arithmetic sequence.
3. I can find the formula for a geometric sequence, find a specified term within a geometric sequence, and find a given number of geometric means between two terms in a geometric sequence.
4. I can write series in expanded form and in summation (sigma) notation.
5. I can find partial sums of arithmetic and geometric series.

Common Core State Standards that are addressed in this unit include: A.SSE.1a, A.SSE.4b, A.CED.2a
For more information see www.corestandards.org/Math/

Introduction to Sequences

In this, the last unit, we will study sequences and series. Sequences are simply a type of pattern.

Definition: A **Sequence** is a function whose domain is the set of natural (counting) numbers, and whose range is the set of term values.

Example: Consider the sequence 3, 5, 7, 9, … can be written in a table as

Term number (x)	1	2	3	4	…
Term value (y)	3	5	7	9	…

The notation that is used is this:

$t_1 = 3$ $t_2 = 5$ $t_3 = 7$ $t_4 = 9$ $t_n = 2n+1$

There are many special types of sequences. We will mainly concentrate on two.

Arithmetic Sequence – all terms are separated by a common difference, d. ← Adding or Subtracting ONLY!

Example: $-2, 2, 6, 10, \ldots$ OR $15, 12, 9, 6, 3, \ldots$

$d = 4$ $d = -3$

Geometric Sequence – all terms are separated by a common ratio, r. ← Multiplication or Division ONLY!

Example: $5, -15, 45, -135$ OR $125, 25, 5, \frac{1}{5}, \ldots$

$r = -3$ $r = \frac{1}{5}$

Break for Practice:

1. Identify the following as arithmetic, geometric, or neither. Then fill in the missing terms.

a.) 7, 12, 17, 22, 27, 32

Arithmetic; $(d = 5)$

b.) 2, -4, 8, -16, 32, -64

Geometric; $(r = -2)$

c.) $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}$

Neither; (denominator is going up by 1)

d.) -1, 2, -3, 4, -5, 6

Neither; (Increasing by 1 with alternating signs)

e.) 21, 15, 9, 3, -3, -9

Arithmetic; $(d = -6)$

f.) 5, 15, 45, 135, 405, 1215

Geometric; $(r = 3)$

2. Find the first 4 terms, and identify the sequence as arithmetic, geometric, or neither.

a.) $t_n = 1 - 2n$ $t_1 = 1-2(1)$ $t_2 = 1-2(2)$ $t_3 = 1-2(3)$ $t_4 = 1-2(5)$

Arithmetic ($d=-2$) $t_1 = -1$ $t_2 = -3$ $t_3 = -5$ $t_4 = -7$

b.) $t_n = 2(3^n)$ $t_1 = 2(3^1)$ $t_2 = 2(3^2)$ $t_3 = 2(3^3)$ $t_4 = 2(3^4)$

Geometric ($r=3$) $t_1 = 6$ $t_2 = 18$ $t_3 = 54$ $t_4 = 162$

c.) $t_n = \frac{1}{n^2}$ $t_1 = \frac{1}{1^2}$ $t_2 = \frac{1}{2^2}$ $t_3 = \frac{1}{3^2}$ $t_4 = \frac{1}{4^2}$

Neither $t_1 = 1$ $t_2 = \frac{1}{4}$ $t_3 = \frac{1}{9}$ $t_4 = \frac{1}{16}$

3. Find the next two terms by looking at the pattern in the difference between terms.

a.) 8, 9, 11, 14, 18, 23

1 2 3 4 5

b.) 5, 7, 11, 17, 25, 35

2 4 6 8 10

Neither; Not adding by the same #.

Extended Practice: Identify the following as arithmetic, geometric, or neither. Then fill in the missing terms.

1. 20, 17,14, 11, 8, 5 Arithmetic → $d=-3$	2. 5, 9, 13, 17, 21, 25 Arithmetic → $d=4$
3. 1, 5, 25, 125, 625, 3125 Geometric → $r=5$	4. 256, 64, 16, 4, 1, $\frac{1}{4}$ Geometric → $r=\frac{1}{4}$
5. 18, 22, 26, 30, 34, 38 Arithmetic → $d=4$	6. 4, 0, -4, -8, -12, -16 Arithmetic → $d=-4$
7. 1, $\frac{1}{4}$, $\frac{1}{9}$, $\frac{1}{16}$, $\frac{1}{25}$, $\frac{1}{36}$ Neither → $\frac{1}{\text{perfect square}}$	8. 32, -16, 8, -4, 2, -1 Geometric → $r=-\frac{1}{2}$

Find the first four terms of the sequence with the given formula. Then tell whether the sequence is arithmetic, geometric, or neither.

9. $t_n = 4n + 3$ 7, 11, 15, 19 Arithmetic d=4	10. $t_n = 2n + 1$ 3, 5, 7, 9 Arithmetic → d=2
11. $t_n = 3^{n-1}$ 1, 3, 9, 27 Geometric → r=3	12. $t_n = 2 \cdot 3^n$ 6, 18, 54, 162 Geometric → r=3
13. $t_n = \frac{(-2)^n}{8}$ $-\frac{1}{4}$, $\frac{1}{2}$, -1, 2 Geometric → r=-2	

Find the next two terms of each sequence by using the pattern in the differences between terms.

14. 60, 48, 38, 30, 24, 20, 18	15. 24, 23, 21, 17, 9, -7, -39
16. 1, 3, 7, 15, 31, 63, 127	17. 0, 1, 4, 13, 40, 121, 364
18. 1, 1, 2, 3, 5, 8, 13, 21, 34	19. 1, 3, 6, 11, 19, 31, 48, 71

Arithmetic Sequences

In this section we shall see how we can write and use formulas for arithmetic sequences.

Consider the sequence 3, 10, 17, 24, … Verify this is arithmetic and identify the common difference.

$t_1 = 3$ $t_2 = 10$ $t_3 = 17$ $t_4 = 24$ $t_n = 3+(n-1)7$

Result: For an arithmetic sequence, the formula is $t_n =$ $t_1+(n-1)d$

$t_1 =$ 1st term, $n =$ term #, $d =$ difference $t_n =$ value of term

Break for Practice:

1. Write a formula for each of the following.

a.) 7, 15, 23, 31, … $t_n = 7+(n-1)8$

+8

$t_1 = 7$

$d = 8$

$t_n = 7+8n-8$

OR

$t_n = 8n-1$

b.) 100, 90, 80, 70, … $t_n = 100+(n-1)(-10)$

$t_1 = 100$

$d = -10$

$t_n = 100-10n+10$

OR

$t_n = -10n+110$

c.) 3, 7, 11, 15, … $t_n = 3+(n-1)7$

$t_1 = 3$

$d = 7$

$t_n = 3+7n-7$

OR

$t_n = 7n-4$

2. Find the specified term in the Arithmetic Sequence.

a.) 2, 5, 8, … $t_{17} =$ 50

$t_1 = 2$

$d = 3$

$n = 17$

$t_{17} = 2+(17-1)3$

$t_{17} = 50$

b.) 912, 882, 852, 822, … $t_{43} =$ -348

$t_1 = 912$

$d = -30$

$n = 43$

$t_{43} = 912+(43-1)(-30)$

$t_{43} = -348$

c.) $t_2 = 9$ $t_5 = 21$ $t_{41} =$ 165

$t_n = t_1+(n-1)d$

$9 = t_1+(2-1)d$

$9 = t_1+d$

$21 = t_1+(5-1)d$

$21 = t_1+4d$

$21 = t_1+4d$

$-\ 9 = t_1+d$

$12 = 3d$

$\frac{12}{3} = \frac{3d}{3}$

$4 = d$

$9 = t_1+4$

$-4 \quad -4$

$5 = t_1$

$t_{41} = 5+(41-1)4$

$\boxed{t_{41} = 165}$

d.) $t_{10} = 41$ $t_{15} = 61$ $t_3 =$ 13

$41 = t_1+(10-1)d$

$41 = t_1+9d$

$61 = t_1+(15-1)d$

$61 = t_1+14d$

$61 = t_1+14d$

$-\ 41 = t_1+9d$

$20 = 5d$ $d = 4$

$61 = t_1+14(4)$

$61 = t_1+56$

$-56 \quad -56$

$t_1 = 5$

$t_3 = 5+(3-1)4$

$\boxed{t_3 = 13}$

Extended Practice: Write a formula for each of the following.

1. 24, 32, 40, 48, … $t_n = 24 + (n-1)8$ or $t_n = 16 + 8n$	2. 30, 20, 10, 0, … $t_n = 30 + (n-1)(-10)$ OR $t_n = 40 - 10n$
3. -3, -10, -17, -24, … $t_n = -3 + (n-1)(-7)$ or $t_n = 4 - 7n$	4. -6, -1, 4, 9, … $t_n = -6 + (n-1)5$ OR $t_n = -11 + 5n$
5. 7, 11, 15, 19, … $t_n = 7 + (n-1)4$ OR $t_n = 3 + 4n$	

Find the specified term of each arithmetic sequence.

1. 4, 9, 14, 19, … (+5) t_{21} = 104 $t_1 = 4$ $d = 5$ $n = 21$ $t_{21} = 4 + (21-1)5$ $t_{21} = 104$	2. 3, 11, 19, … (+8) t_{31} = 243 $t_1 = 3$ $d = 8$ $n = 31$ $t_{31} = 3 + (31-1)8$ $t_{31} = 243$
3. 100, 98, 96, … (-2) t_{25} = 52 $t_1 = 100$ $d = -2$ $n = 25$ $t_{25} = 100 + (25-1)(-2)$ $t_{25} = 52$	4. 3, 3.5, 4, 4.5, … (+0.5) t_{101} = 53 $t_1 = 3$ $d = 0.5$ $n = 101$ $t_{101} = 3 + (101-1)(0.5)$ $t_{101} = 53$
5. 17, 7, -3, … (-10) t_{1000} = -9973 $t_1 = 17$ $d = -10$ $n = 1000$ $t_{1000} = 17 + (1000-1)(-10)$ $t_{1000} = -9973$	6. $t_2 = 7$ $t_4 = 8$ t_1 = 6.5 $7 = t_1 + (2-1)d \rightarrow 7 = t_1 + d \rightarrow 7 = t_1 + \frac{1}{2}$ $8 = t_1 + (4-1)d \rightarrow 8 = t_1 + 3d$ $-1 = -2d$ $\frac{-1}{-2} = \frac{-2d}{-2}$ $d = \frac{1}{2}$ $-\frac{1}{2} \quad -\frac{1}{2}$ $t_1 = 6.5$
7. $t_8 = 60$ $t_{12} = 48$ t_{40} = -36 $48 = t_1 + (12-1)d \rightarrow 48 = t_1 + 11d$ $60 = t_1 + (8-1)d \rightarrow 60 = t_1 + 7d$ $-12 = 4d$ $\frac{-12}{4} = \frac{4d}{4}$ $d = -3$ $\rightarrow 60 = t_1 + (-3)(7)$ $60 = t_1 - 21$ $+21 \quad +21$ $81 = t_1$ $t_{40} = 81 + (40-1)(-3)$ $\boxed{t_{40} = -36}$	

Now let's look at a couple of other types of problems that we can solve with the arithmetic sequence formula.

Break for Practice:

1. Find the position, n of the underlined term in each arithmetic sequence.

a.) 5, 8, 11, 14, … , <u>68</u> , …

$+3$

$t_1 = 5$

$d = 3$

$t_n = 68$

$68 = 5 + (n-1)3$

$-5 \quad -5$

$\frac{63}{3} = \frac{3(n-1)}{3}$

$21 = n - 1$

$+1 \quad +1$

$n = 22$

b.) 88, 83, 78, 73, … , <u>13</u> , …

$t_1 = 88$

$d = -5$

$t_n = 13$

$13 = 88 + (n-1)(-5)$

$-88 \quad -88$

$\frac{-75}{-5} = \frac{-5(n-1)}{-5}$

$15 = n - 1$

$+1 \quad +1$

$n = 16$

The other idea we will work on is that of finding arithmetic means.

Arithmetic Means – terms between two given terms in an arithmetic sequence.

2. Find the stated number of arithmetic means between the two given terms.

a.) Two between 8 and 35.

8, __, __, 35

$t_1 \qquad t_4$

$t_1 = 8$

$t_4 = 35$

$n = 4$

$35 = 8 + (4-1)d$

$-8 \quad -8$

$\frac{27}{3} = \frac{3d}{3}$

$9 = d$

$\boxed{8, \underline{17}, \underline{26}, 35}$

b.) Five between 83 and -25.

83, __, __, __, __, __, -25

$t_1 \qquad t_7$

$t_1 = 83$

$t_7 = -25$

$n = 7$

$-25 = 83 + (7-1)d$

$-83 \quad -83$

$\frac{-108}{6} = \frac{6d}{6}$

$-18 = d$

$\boxed{83, \underline{65}, \underline{47}, \underline{29}, \underline{11}, \underline{-7}, -25}$

c.) One between -7.8 and 3.6.

$\frac{-7.8 + 3.6}{2} = \frac{-4.2}{2} = -2.1$

Extended Practice: Find the position, n, of the underlined term in each arithmetic sequence.

1. 25, 33, 41, …, <u>145</u> ,…	2. 40, 37, 34, …, <u>-29</u>, …
$t_1 = 25$ $d = 8$ $t_n = 145$ $145 = 25 + (n-1)8$ $-25 \quad -25$ $\frac{120}{8} = \frac{8(n-1)}{8}$ $15 = n - 1$ $+1 \quad +1$ $16 = n$	$t_1 = 40$ $d = -3$ $t_n = -29$ $-29 = 40 + (n-1)(-3)$ $-40 \quad -40$ $\frac{-69}{-3} = \frac{-3(n-1)}{-3}$ $23 = n - 1$ $+1 \quad +1$ $24 = n$

Find the stated number of arithmetic means between the two given terms.

3. One between -3 and 7

$\frac{-3+7}{2} = \frac{4}{2} = 2$

$\boxed{-3, \underline{2}, 7}$

4. One between 2.3 and 9.1

$\frac{2.3+9.1}{2} = \frac{11.4}{2} = 5.7$

$\boxed{2.3, \underline{5.7}, 9.1}$

5. Two between 15 and 45

$t_1 = 15$

$t_4 = 45$

$n = 4$

15, ___, ___, 45

t_1 t_4

$45 = 15 + (4-1)d$

$-15 \quad -15$

$30 = 3d$

$\frac{30}{3} = \frac{3d}{3}$

$10 = d$

$\boxed{15, \underline{25}, \underline{35}, 45}$

6. Four between 15 and 45

$t_1 = 15$

$t_6 = 45$

$n = 6$

15, ___, ___, ___, ___ 45

t_1 t_6

$45 = 15 + (6-1)d$

$-15 \quad -15$

$30 = 5d$

$\frac{30}{5} = \frac{5d}{5}$

$6 = d$

$\boxed{15, \underline{21}, \underline{27}, \underline{33}, \underline{39}, 45}$

7. How many terms are in the sequence 18, 24, 30, … , 618?

$+6$

$t_1 = 18$

$d = 6$

$t_n = 618$

$618 = 18 + (n-1)6$

$-18 \quad -18$

$600 = 6(n-1)$

$\frac{600}{6} = \frac{6(n-1)}{6}$

$100 = n-1$

$+1 \quad +1$

$\boxed{n = 101}$

8. How many terms are in the sequence 44, 36, 28, … , -380?

-8

$t_1 = 44$

$d = -8$

$t_n = -380$

$-380 = 44 + (n-1)(-8)$

$-44 \quad -44$

$-424 = -8(n-1)$

$\frac{-424}{-8} = \frac{-8(n-1)}{-8}$

$53 = n-1$

$+1 \quad +1$

$\boxed{n = 54}$

Geometric Sequences

Now that we have spent time with arithmetic sequences, we will switch our focus to geometric sequences.

Consider the sequence 3, 6, 12, 24, … Verify this is geometric and identify the common ratio.

$t_1 = 3$ $t_2 = 6$ $t_3 = 12$ $t_4 = 24$ $t_n = 3 \cdot 2^{n-1}$

↑

Result: For a geometric sequence, the formula is $t_n = t_1 \cdot r^{n-1}$

$t_1 = 1^{st}$ term, r = ratio n = term # t_n = value of term.

Break for Practice:

1. Write a formula for each of the following.

a.) 1000, 200, 40, 8, …

$\div 5$

$t_1 = 1000$

$r = 1/5$

$\boxed{t_n = 1000\left(\frac{1}{5}\right)^{n-1}}$

b.) -100, 50, -25, 12.5, …

$-1/2$

$t_1 = -100$

$r = -1/2$

$\boxed{t_n = -100\left(-\frac{1}{2}\right)^{n-1}}$

2. Find the specified term in each geometric sequence.

a.) 2048, 1024, 512, … $t_{20} =$ 0.00390625

$t_1 = 2048$

$r = \frac{1024}{2048} = \frac{1}{2}$

$n = 20$

$t_{20} = 2048 \cdot \frac{1}{2}^{20-1}$

$t_{20} = 0.00390625$

b.) 6, 9, 13.5, … $t_{10} =$ 203.6601563

$t_1 = 6$

$r = \frac{9}{6} = \frac{3}{2}$

$n = 10$

$t_{10} = 6 \cdot \frac{3}{2}^{10-1}$

$t_{10} = 203.6601563$

c.) $t_2 = 6$ $t_7 = 192$ $t_{12} =$ 6144

$t_n = t_1 \cdot r^{n-1}$

$6 = t_1 \cdot r^{2-1}$

$6 = t_1 \cdot r$

$192 = t_1 \cdot r^{7-1}$

$192 = t_1 \cdot r^6$

$\frac{192 = t_1 \cdot r^6}{6 = t_1 \cdot r}$ → $6 = t_1 \cdot 2^1$

$\sqrt[5]{32} = \sqrt[5]{r^5}$

$r = 2$

$\frac{6}{2} = \frac{2t_1}{2}$

$3 = t_1$

$t_{12} = 3 \cdot 2^{12-1}$

$t_{12} = 6144$

Extended Practice: Write a formula for each of the following.

1. 2, 6, 18, 54, … $\times 3$ $t_1 = 2$ $r = 3$ $t_n = 2 \cdot 3^{n-1}$	2. 500, 100, 20, 4, … $\div 5$ $t_1 = 500$ $r = 1/5$ $t_n = 500 \cdot (1/5)^{n-1}$
3. 64, -48, 36, -27, … $t_1 = 64$ $r = \frac{-48}{64} = -\frac{6}{8} = -\frac{3}{4}$ $t_n = 64 \cdot \left(-\frac{3}{4}\right)^{n-1}$	

Find the specified term of each geometric sequence.

4. 2, 6, 18, 54, … t_{10} = 39366 $\times 3$ $t_1 = 2$ $r = 3$ $n = 10$ $t_{10} = 2 \cdot 3^{10-1}$ $t_{10} = 2 \cdot 3^9$ $t_{10} = 39366$	5. 5, 10, 20, 40, … t_{12} = 10,240 $\times 2$ $t_1 = 5$ $r = 2$ $n = 12$ $t_{12} = 5 \cdot 2^{12-1}$ $t_{12} = 5 \cdot 2^{11}$ $t_{12} = 10240$
6. 40, -20, 10, -5, … t_{11} = 0.0390625 $\div -2$ $t_1 = 40$ $r = -\frac{1}{2}$ $n = 11$ $t_{11} = 40 \cdot \left(-\frac{1}{2}\right)^{11-1}$ $t_{11} = 40 \cdot \left(-\frac{1}{2}\right)^{10}$ $t_{11} = 0.0390625$	7. -10, 50, -250, 1250, … t_9 = -3906250 $\times -5$ $t_1 = -10$ $r = -5$ $n = 9$ $t_9 = -10 \cdot (-5)^{9-1}$ $t_9 = -10 \cdot (-5)^8$ $t_9 = -3906250$