3. $(\sqrt{6}-2)^2 \rightarrow (\sqrt{6}-2)(\sqrt{6}-2)$

$= \sqrt{36} - 2\sqrt{6} - 2\sqrt{6} + 4$

$= 6 - 4\sqrt{6} + 4$

$= 10 - 4\sqrt{6}$

4. $(2+\sqrt{3})(2-\sqrt{3}) = 4 - \cancel{2\sqrt{3}} + 2\sqrt{3} - \sqrt{9}$

$= 4 - 3$

$= 1$

5. $(1+\sqrt{5})(1-\sqrt{5}) = 1 - \cancel{\sqrt{5}} + \sqrt{5} - \sqrt{25}$

$= 1 - 5$

$= -4$

The last two examples illustrate multiplying Conjugates. Conjugates are identical binomials except for the middle sign (They are opposite). Note that when these are multiplied, the radical disappears. This idea can be used to rationalize denominators in the following problems.

Break for Practice: Simplify

1. $\frac{5}{(2-\sqrt{3})} \frac{(2+\sqrt{3})}{(2+\sqrt{3})} = \frac{10+5\sqrt{3}}{4+\cancel{2\sqrt{3}}-2\sqrt{3}-\sqrt{9}}$

$= \frac{10+5\sqrt{3}}{4-3}$

$= 10+5\sqrt{3}$

2. $\frac{3}{(\sqrt{5}+\sqrt{3})} \frac{(\sqrt{5}-\sqrt{3})}{(\sqrt{5}-\sqrt{3})} = \frac{3\sqrt{5}-3\sqrt{3}}{\sqrt{25}-\cancel{\sqrt{15}}+\cancel{\sqrt{15}}-\sqrt{9}}$

$= \frac{3\sqrt{5}-3\sqrt{3}}{5-3}$

$= \frac{3\sqrt{5}-3\sqrt{3}}{2}$

Extended Practice: Simplify

1. $(3+\sqrt{7})(3-\sqrt{7}) = 2$	2. $(\sqrt{7}+1)^2 = 8 + 2\sqrt{7}$

3. $(6-\sqrt{3})(4+\sqrt{3}) = 21 + 2\sqrt{3}$

4. $\frac{1}{4-\sqrt{3}} = \frac{4+\sqrt{3}}{13}$

5. $\frac{1}{6+\sqrt{3}} = \frac{6-\sqrt{3}}{33}$

6. $(3+4\sqrt{3})(2-\sqrt{3}) = -6 + 5\sqrt{3}$

7. $\frac{3}{\sqrt{5}+\sqrt{2}} = \sqrt{5} - \sqrt{2}$

8. $\frac{\sqrt{5}+1}{\sqrt{5}-3} = -2 - \sqrt{5}$

Solving Equations Containing Radicals

In this section we will learn how to solve radical equations. Radical equations include a variable under a radical sign. An example of a radical equation is the formula for calculating braking distance for a moving car. **The formula is $s = \sqrt{22d}$**. s is the speed in miles per hour, and d is the braking distance in feet.

What was the speed of a car that left skid marks 160 feet long? $\Rightarrow$ $S = \sqrt{22(160)}$

$S = \sqrt{3520}$

$S \approx 59.33$ mph

What braking distance is needed for stopping a car travelling 65 mph? $\Rightarrow$ $(65) = (\sqrt{22d})^2$

$\frac{4225}{22} = \frac{22d}{22}$

192.05 ft $\approx d$

This shows an example of solving a very simple radical equation. In general, these are the steps for solving radical equations.

Steps:

1. Isolate a radical term.
2. Undo the radical.
3. Repeat if necessary until all radicals are gone.
4. Solve the resulting equation.
5. Check solutions to eliminate any extraneous solutions.

Break for Practice: Solve each radical equation and identify any extraneous solutions.

1. $(\sqrt{x-5})^2 = (5)^2 \rightarrow x - 5 = 25$

$+5 \quad +5$

$x = 30$

ck $\sqrt{25} = 5$ ✓

2. $(\sqrt{3x-5})^2 = (4)^2 \rightarrow 3x - 5 = 16$

$+5 \quad +5$

$\frac{3x}{3} = \frac{21}{3}$

$x = 7$

ck $\sqrt{16} = 4$ ✓

3. $\sqrt[3]{4x} + 7 = 5 \rightarrow (\sqrt[3]{4x})^3 = (-2)^3$

$-7 \quad -7$

$\frac{4x}{4} = \frac{-8}{4}$

$x = -2$

ck $\sqrt[3]{-8} + 7 = 5$

$-2 + 7 = 5$ ✓

4. $(\sqrt[3]{x-2})^3 = (2)^3 \rightarrow x - 2 = 8$

$+2 \quad +2$

$x = 10$

ck $\sqrt[3]{8} = 2$ ✓

5. $3 = x + \sqrt{x-1}$

$-x \quad -x$

$(3-x)^2 = (\sqrt{x-1})^2$

$3-x)(3-x) = x - 1$

$9 - 3x - 3x + x^2 = x - 1$

$x^2 - 6x + 9 = x - 1$

$-x \quad +1 \quad -x \quad +1$

$x^2 - 7x + 10 = 0$

$(x-2)(x-5) = 0$

$x = 2, \quad x = \cancel{5}$

ck $3 = 2 + \sqrt{1}$

$3 = 2 + 1$ ✓

ck $5 = 5 + \sqrt{4}$

$5 = 5 + 2$ ✗

6. $\sqrt{x+6} - x = 4$

$+x \quad +x$

$(\sqrt{x+6})^2 = (x+4)^2$

$x + 6 = (x+4)(x+4)$

$x + 6 = x^2 + 4x + 4x + 16$

$x + 6 = x^2 + 8x + 16$

$-x \quad -6 \quad -x \quad -6$

$0 = x^2 + 7x + 10$

$0 = (x+2)(x+5)$

$x = -2, \quad x = \cancel{-5}$

ck: $\sqrt{4} + 2 = 4$

$2 + 2 = 4$ ✓

ck: $\sqrt{1} + 5 = 4$

$1 + 5 = 4$ ✗

Extended Practice: Solve each radical equation and identify any extraneous solutions.

1. $\sqrt{4x-3} = 5$ $x = 7$

2. $\sqrt{3n+1} = 7$ $n = 16$

3. $\sqrt[3]{3m+1} = 4$ $m = 21$

4. $7 - \sqrt[3]{9c} = 4$ $c = 3$

5. $\sqrt{x+2} = x$ $x = 2$

6. $\sqrt{2n+3} = n$ $n = 3$

7. $5 + \sqrt{a+7} = a$ $a = 9$

8. $\sqrt{2x+5} - 1 = x$ $x = 2$

9. If you are near the top of a tall building on a clear day, how far can you see? If a building is h feet high, then the distance d (in miles) to the earth's horizon is approximately $d = \sqrt{\frac{3}{2}h}$

a) The observatory of a tall building in Chicago is 607 feet high. What is the distance to the horizon from this observatory?

$\underbrace{d \approx 30.2 \text{ miles}}$

b) Solve the formula for h.

$\underbrace{h \approx \frac{2}{3}d^2}$

10. If a pendulum is l cm long, then the time T (in seconds) that it takes the pendulum to swing back and forth once is given by $T = 2\pi\sqrt{\frac{l}{g}}$, where $\pi \approx 3.14$ and $g \approx 980$.

a) Find the value of T if the pendulum is 20 cm long.

$\underbrace{T \approx 0.9 \text{ seconds}}$

b) Solve the formula for l in terms of T, g, and π.

$\underbrace{l = \frac{T^2 g}{4\pi 2}}$

All of the previous problems had only one radical. Now we shall see how to use our steps for solving when there are multiple radicals in the equations.

Break for Practice: Solve each radical equation and identify any extraneous solutions.

1. $\sqrt{2x-4} - \sqrt{x-3} = 1 \rightarrow (\sqrt{2x-4})^2 = (1+\sqrt{x-3})^2 \rightarrow (1+\sqrt{x-3})(1+\sqrt{x-3})$

$+\sqrt{x-3}$

$2x-4 = 1+\sqrt{x-3}+\sqrt{x-3}+x-3$

$2x-4 = 1+2\sqrt{x-3}+x-3$

$2x-4 = x-2+2\sqrt{x-3}$

$-x \quad +2 \qquad -x \quad +2$

$(x-2)(x-2) \leftarrow (x-2)^2 = (2\sqrt{x-3})^2$

$x^2-2x-2x+4 = 4(x-3)$

$x^2-4x+4 = 4x-12$

$-4x \quad +12 \qquad -4x \quad +12$

$x^2-8x+16 = 0$

$(x-4)(x-4) = 0$

$x = 4$

CK: $\sqrt{4} - \sqrt{1} = 1$

$2-1=1$ ✓

2. $2\sqrt{x+4} - \sqrt{2x+25} = 1 + \sqrt{2x+25} \Rightarrow \left(2\sqrt{x+4}\right)^2 = \left(1 + \sqrt{2x+25}\right)^2 \rightarrow \left(1+\sqrt{2x+25}\right)\left(1+\sqrt{2x+25}\right)$

$$4(x+4) = 1 + \sqrt{2x+25} + \sqrt{2x+25} + 2x+25$$

$$4x+16 = 2x+26 + 2\sqrt{2x+25}$$

$$-2x \quad -26 \quad -2x \quad -26$$

$$(2x-10)(2x-10) \qquad \leftarrow (2x-10)^2 = \left(2\sqrt{2x+25}\right)^2$$

$$4x^2 - 20x - 20x + 100 = 4(2x+25)$$

$$4x^2 - 40x + 100 = 8x + 100$$

$$-8x \quad -100 \quad -8x \quad -100$$

$$4x^2 - 48x = 0$$

$$4x(x-12) = 0$$

$$\frac{4x}{4} = \frac{0}{4} \qquad x - 12 = 0$$

$$\cancel{x=0} \qquad x = 12$$

CK: $2\sqrt{4} - \sqrt{25} = 1$

$4 - 5 = 1$ ✗

CK: $2\sqrt{16} - \sqrt{49} = 1$

$8 - 7 = 1$ ✓

Extended Practice: Solve each radical equation and identify any extraneous solutions.

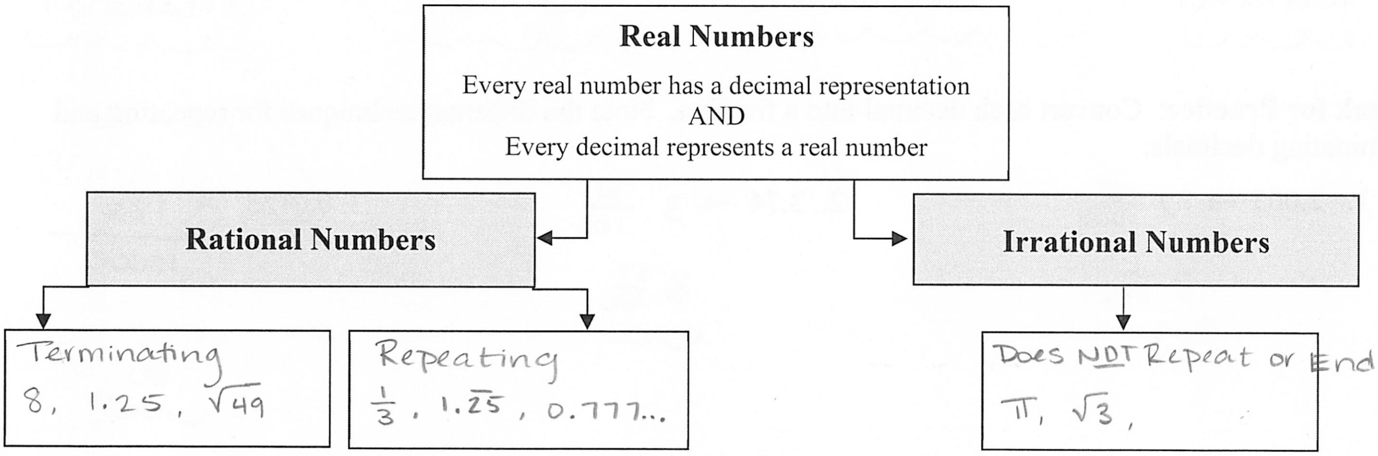
1. $\sqrt{y} + \sqrt{y+5} = 5$ $\qquad y = 4$

2. $\sqrt{x-7} + \sqrt{x} = 7$ $\qquad x = 16$

3. $\sqrt{3a-2}-\sqrt{2a-3}=1$ $x=2,\ x=6$

Rational and Irrational Numbers

In this section we will explore the differences of rational and irrational numbers, and learn several methods for converting between fractions and decimals. Since we will be concentrating on numbers written in decimal form, we will need to know when decimals are rational or irrational.



Break for Practice: Convert the following rationals to decimal form.

1. $\frac{7}{8}$ → $8\overline{)7.000}$ quotient 0.875

64

60

-56

-40

40

0

$= 0.875$

Terminating

2. $\frac{7}{3}$ → $3\overline{)7.000}$ quotient 2.333

-6

10

9

10

9

10

$= 2.\overline{3}$

Repeating

3. $\frac{9}{11} \rightarrow$

$$\begin{array}{r} 0.8181 \\ 11\overline{)90000} \\ -88 \\ \hline 20 \\ 11 \\ \hline 90 \\ 88 \\ \hline 20 \\ 11 \\ \hline 9 \end{array}$$

$= 0.\overline{81}$

Repeating

4. $\frac{13}{25} \rightarrow$

$$\begin{array}{r} 0.52 \\ 25\overline{)1300} \\ -125 \\ \hline 50 \\ -50 \\ \hline 0 \end{array}$$

$= 0.52$

Terminating

Break for Practice: Classify each number as rational (R) or irrational (I).

1. $0.\overline{48}$

Rational

2. 1.481481148111 ...

Irrational

3. $\sqrt{3} + \sqrt{27} = 3 + 3\sqrt{3}$

Irrational

4. $\sqrt{3} \cdot \sqrt{27} = \sqrt{81}$

$= 9$

Rational

5. $\frac{\pi}{2}$

Irrational

6. $\sqrt{\frac{25}{64}} = \frac{5}{8} = 0.625$

Rational

7. $\sqrt{\frac{1}{10}} = \frac{1}{\sqrt{10}}$

Irrational

Break for Practice: Convert each decimal into a fraction. Note the different techniques for repeating and terminating decimals.

1. 2.005 → $2\frac{5}{1000}$

$2\frac{1}{200}$

2. 3.74 → $3\frac{74}{100}$

$3\frac{37}{50}$

3. 0.0125 → $\frac{125}{10000}$

$= \frac{1}{80}$

<u>Steps for writing a repeating decimal as a fraction:</u>

1. Let $N =$ the number
2. Multiply the given number by 10^n
   (n =the number of digits **<u>under</u>** the block of repeating digits.
3. Subtract the original N (Step#1) → this will eliminate the repeating block.
4. Solve of N and get in Simplest form.

4. $0.\overline{08}$

$n=2 \rightarrow 10^2=100$

② $100\,(① N) = (0.0\overline{8})\,100$

$100N = 8.\overline{8}$

③ $- \quad N \quad -0.\overline{8}$

$99N = 8$

④ $\frac{99N}{99} = \frac{8}{99}$

$N = \frac{8}{99}$

5. $0.8\overline{7}$

$n=1 \rightarrow 10^1=10$

② $10\,(① N) = (0.8\overline{7})\,10$

$10N = \overset{7}{8}.\overset{17}{\overline{7}}\overline{7}$

③ $- \quad N \quad - 0.8\overline{7}$

$9N = 7.9$

$\frac{9N}{9} = \frac{7.9}{9}$

④ $N = \frac{7.9}{90}$

$N = \frac{79}{90}$

6. $0.\overline{9}$

$n=1 \rightarrow 10^1=10$

② $10\,(① N) = (0.\overline{9})\,10$

$10N = 9.\overline{9}$

$- \quad N \quad -0.\overline{9}$

$9N = 9$

$\frac{9N}{9} = \frac{9}{9}$

$N = 1$

Extended Practice:
Classify each real number or expression as either rational (R), or irrational (I).

1. $\sqrt{49}$ Rational	2. $\sqrt{50}$ Irrational	3. π Irrational	4. $\frac{22}{7}$ Irrational
5. $\pi + \frac{1}{\pi}$ Irrational	6. $\pi \cdot \frac{1}{\pi}$ Rational	7. $\sqrt{\frac{1}{2}} + \sqrt{\frac{1}{8}}$ Irrational	8. $\sqrt{\frac{1}{2}} \cdot \sqrt{\frac{1}{8}}$ Rational
9. 1.23 Rational	10. $1.\overline{23}$ Rational	11. 1.2345678910111213 ... Irrational	

Write each fraction as a repeating or terminating decimal.

12. $\frac{5}{8} = 0.625$	13. $\frac{5}{11} = 0.\overline{45}$	14. $\frac{13}{7} = 1.\overline{857142}$	15. $\frac{13}{4} = 3.25$

Write each decimal as a fraction in lowest terms.

16. 3.004 $= \frac{751}{250}$ OR $3\ ^{1}/_{250}$	17. 4.72 $= \frac{118}{25}$ OR $4\frac{18}{25}$	18. 0.1375 $= \frac{11}{80}$

Write each decimal as a fraction in lowest terms.

19. $0.\overline{5}$ $= \frac{5}{9}$	20. $0.\overline{36}$ $= \frac{4}{11}$	21. $1.\overline{27}$ $= \frac{14}{11}$

The Imaginary Number, *i*

In this section we shall see how we can simplify expressions involving square roots of negatives. Up until this time we simply said that they were not real. Instead they are imaginary. Imaginary numbers have applications in electricity, optics, hydrodynamics, etc.

Definition: $i = \sqrt{-1}$ *and* $i^2 = -1$

By extension, we can say that $\sqrt{-r} =$ $i\sqrt{r}$ This allows us to solve and simplify many more algebraic equations and expressions.

Numbers in the form b*i* , where b does not equal zero, are called pure imaginary.

Break for Practice: Simplify.

1. $\sqrt{-49} \rightarrow i\sqrt{49}$ $= 7i$

2. $\sqrt{-10} \rightarrow i\sqrt{10}$

3. $\sqrt{-28} \rightarrow i\sqrt{28} \begin{array}{l}\sqrt{4}\\ \sqrt{7}\end{array}$ $= 2i\sqrt{7}$

4. $-3\sqrt{-144} \rightarrow -3i\sqrt{144}$

$= -3i(12)$

$= \underline{-36i}$

5. $3\sqrt{-12} = 3i\sqrt{12}$ $\begin{smallmatrix} \sqrt{4} \\ \sqrt{3} \end{smallmatrix}$

$= 3i(2)\sqrt{3}$

$= \underline{6i\sqrt{3}}$

Simplify the product.

6. $7i \cdot 5i = 35i^2$

$= 35(-1)$

$= \underline{-35}$

7. $(6i)^2 = 36i^2$

$= 36(-1)$

$= \underline{-36}$

8. $\left(2i\sqrt{3}\right)^2 = 4i^2(3)$

$= (-1)(12)$

$= \underline{-12}$

9. $\sqrt{-4} \cdot \sqrt{-25} \rightarrow i\sqrt{4} \cdot i\sqrt{25}$

$= i^2\sqrt{100}$

$= -1(10)$

$= \underline{-10}$

10. $\sqrt{7} \cdot \sqrt{-14} \rightarrow \sqrt{7} \cdot i\sqrt{14}$

$= i\sqrt{98}$ $\begin{smallmatrix} \sqrt{49} \\ \sqrt{2} \end{smallmatrix}$

$= \underline{7i\sqrt{2}}$

11. $\sqrt{-6} \cdot \sqrt{-15} \rightarrow i\sqrt{6} \cdot i\sqrt{15}$

$= i^2\sqrt{90}$ $\begin{smallmatrix} \sqrt{9} \\ \sqrt{10} \end{smallmatrix}$

$= (-1)3\sqrt{10}$

$= \underline{-3\sqrt{10}}$

Simplify the sum or difference.

12. $\sqrt{-20} + \sqrt{-45}$

$\sqrt{4}\sqrt{5}$ $\quad$ $\sqrt{9} \cdot \sqrt{5}$

$= i\sqrt{20} + i\sqrt{45}$

$= 2i\sqrt{5} + 3i\sqrt{5}$

$= \underline{5i\sqrt{5}}$

13. $4\sqrt{-3} - \sqrt{-75}$

$= 4i\sqrt{3} - i\sqrt{75}$ $\begin{smallmatrix} \sqrt{25} \\ \sqrt{3} \end{smallmatrix}$

$= 4i\sqrt{3} - 5i\sqrt{3}$

$= \underline{-i\sqrt{3}}$

Divide. (Note: Just like radicals, we can't leave an *i* in the denominator)

14. $\frac{5 \cdot i}{i \cdot i} = \frac{5i}{i^2}$

$= \frac{5i}{-1}$

$= \underline{-5i}$

15. $\frac{6 \cdot i}{7i \cdot i} = \frac{6i}{7(i^2)}$

$= \frac{6i}{7(-1)}$

$= \underline{\frac{6i}{-7}}$

16. $\frac{10}{\sqrt{-5}} \rightarrow \frac{10}{i\sqrt{5}} \cdot \frac{i\sqrt{5}}{i\sqrt{5}}$

$= \frac{10i\sqrt{5}}{i^2\sqrt{25}}$

$= \frac{10i\sqrt{5}}{(-1)(5)}$

$= \frac{10i\sqrt{5}}{-5} \rightarrow \underline{-2i\sqrt{5}}$

Solve

17. $x^2 + 100 = 0$

$-100 \quad -100$

$\sqrt{x^2} = \sqrt{-100}$

$\underline{x = \pm 10i}$

18. $3x^2 + 23 = 5$

$-23 \quad -23$

$\frac{3x^2}{3} = \frac{-18}{3}$

$\sqrt{x^2} = \sqrt{-6}$

$\underline{x = \pm i\sqrt{6}}$

Extended Practice: Simplify

1. $\sqrt{-81}$ $= 9i$

2. $-4\sqrt{-36}$ $= -24i$

3. $\sqrt{-20}$ $= 2i\sqrt{5}$

4. $3\sqrt{-8}$ $= 6i\sqrt{2}$

5. $2i \cdot 3i$ $= -6$

6. $\sqrt{7} \cdot \sqrt{-7}$ $= 7i$

7. $\sqrt{-5} \cdot \sqrt{-10}$ $= -5\sqrt{2}$

8. $(7i)^2$ $= -49$

9. $(-i)^2$ $= -1$

10. $\left(i\sqrt{2}\right)^2$ $= -2$

11. $\frac{8}{3i}$ $= -\frac{8i}{3}$

12. $\frac{21}{\sqrt{-7}}$ $= -3i\sqrt{7}$

Simplify

13. $\sqrt{-25} + \sqrt{-36}$ $= 11i$

14. $3\sqrt{-2} - \sqrt{-50}$ $= -2i\sqrt{2}$

Solve

15. $x^2 + 144 = 0$ $x = \pm 12i$

16. $3u^2 + 40 = 4$ $u = \pm 2i\sqrt{3}$

Complex Numbers

In the last section, we saw what pure imaginary numbers look like. Previously, you already knew what real (anything on the number line) numbers look like. If we put these two sets of numbers together, we form the set of complex numbers. In this section we will learn how to perform the four basic operations on complex numbers.

Definition:

Complex Numbers – numbers in the form $a + bi$ or $a - bi$, where a *and* $b \in reals$.

Examples: $2 + 3i$ *and* $7 - i\sqrt{3}$

Note: $\underset{\uparrow}{a} \pm \underbrace{bi}$ – imaginary#

Real Number

Break for Practice: Simplify the sum or difference.

1. $(7 + 3i) + (2 - 5i) \rightarrow 7 + 3i + 2 - 5i$

$= 9 - 2i$

2. $(4 - 7i) - (5 - 3i) \rightarrow 4 - 7i - 5 + 3i$

$= -1 - 4i$

3. $4(1 - 2i) + 3(-7 + 5i) = 4 - 8i - 21 + 15i$

$= -17 + 7i$

4. $2(-3 + i) - 5(2 - 2i) = -6 + 2i - 10 + 10i$

$= -16 + 12i$

Simplify the product.

5. $2i(4 + 7i) = 8i + 14i^2$

$= 8i + 14(-1)$

$= 8i - 14$

$= -14 + 8i$

6. $-4i(-5 + i) = 20i - 4i^2$

$= 20i - 4(-1)$

$= 20i + 4$

$= 4 + 20i$

7. $(-3 + 2i)(4 + 5i) = -12 - 15i + 8i + 10i^2$

$= -12 - 7i + 10(-1)$

$= -12 - 7i - 10$

$= -22 - 7i$

8. $(3 - 5i)^2 \rightarrow (3 - 5i)(3 - 5i)$

$= 9 - 15i - 15i + 25i^2$

$= 9 - 30i + 25(-1)$

$= 9 - 30i - 25$

$= -16 - 30i$

9. $(2+3i)(2-3i) = 4-6i+6i-9i^2$

$= 4-9(-1)$

$= 4+9$

$= 13$

The last example showed the product of Complex Conjugates. These hold the key for simplifying quotients of complex numbers.

Simplify the quotient.

10. $\frac{6}{(1+3i)}\frac{(1-3i)}{(1-3i)} = \frac{6-18i}{1-3i+3i-9i^2}$

$= \frac{6-18i}{1-9(-1)}$

$= \frac{6-18i}{1+9}$

$= \frac{6-18i}{10} \xrightarrow{(\div 2)} \frac{3}{5}-\frac{9}{5}i$

11. $\frac{(2+i)(6+2i)}{(6-2i)(6+2i)} = \frac{12+4i+6i+2i^2}{36+12i-12i-4i^2}$

$= \frac{12+10i+2(-1)}{36-4(-1)}$

$= \frac{12+10i-2}{36+4}$

$= \frac{10+10i}{40}$ $(\div 10)$

$= \frac{1}{4}+\frac{i}{4}$

Extended Practice: Simplify

1. $(9+2i)+(1-7i) = 10-5i$	2. $(5-7i)-(8+2i) = -3-9i$
3. $3(-2+i)-4(3-2i) = -18+11i$	4. $i(3+4i) = -4+3i$
5. $-4i(-2+i) = 4+8i$	6. $(3-i)(3+i) = 10$

7. $(-4+i)(8+5i) = -37-12i$

8. $(2-4i)^2 = -12-16i$

9. $(-1+i\sqrt{3})^2 = -2-2i\sqrt{3}$

10. $\frac{5}{3+4i} = \frac{3}{5} - \frac{4i}{5}$

11. $\frac{15}{2-i} = 6+3i$

12. $\frac{-1-2i}{-1+2i} = -\frac{3}{5} + \frac{4i}{5}$

Find the reciprocal of each of the following.

13. $2+3i = \frac{2}{13} - \frac{3i}{13}$

14. $1-4i = \frac{1}{17} + \frac{4i}{17}$