Answer Key

Algebra II

Unit 6
Irrational and Complex Numbers

Priority Standards:

NS.A.1:Know that numbers that are not rational are called irrational. Understand informally that every number has a decimal expansion; for rational numbers show that the decimal expansion repeats eventually, and convert a decimal expansion which repeats eventually into a rational number.

N-CN.2:Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

Unit "I can" statements:

1. I can find roots of real numbers.
2. I can simplify expressions involving radicals.
3. I can simplify expressions involving sums of radicals.
4. I can simplify products and quotients of binomials that contain radicals.
5. I can solve equations containing radicals.
6. I can find and use decimal representations of real numbers.
7. I can use imaginary numbers to simplify square roots of negative numbers.
8. I can add, subtract, multiply, and divide complex numbers.

Common Core State Standards that are addressed in this unit include: NS.A.1, N-RN.3, N-CN.1, N-CN.2, N-CN.3, N-CN.7. For more information see www.corestandards.org/Math/

Roots of Real Numbers

In this unit we will be exploring roots and radicals. We will begin with the ones you are familiar with, and extend the ideas from there.

Square Roots are solutions to the equation $x^2 = b$. Every positive number b has 2 square roots, $\sqrt{b}$ and $-\sqrt{b}$. The positive square root, $\sqrt{b}$, is known as the principal square root.

(Note: Since the square of a real number is never negative, the equation $x^2 = b$ has no real solution if $b < 0$.)

Break for Practice: Simplify

1. $\sqrt{25} = 5$
2. $-\sqrt{36} = -6$
3. $\sqrt{0.81} = 0.9$
4. $\sqrt{-25} \rightarrow$ No Solution
5. $\sqrt{\frac{4}{49}} \rightarrow \frac{\sqrt{4}}{\sqrt{49}} = \frac{2}{7}$

Solve each equation. * 2 solutions ⇒ answers need +/-

6. $\sqrt{x^2} = \sqrt{16}$

$x = \pm 4$

7. $x^2 - 9 = 0$

$+9 \quad +9$

$\sqrt{x^2} = \sqrt{9}$

$x = \pm 3$

8. $x^2 + 25 = 0$

$-25 \quad -25$

$\sqrt{x^2} = \sqrt{-25}$

No Solution

10. $3x^2 - 21 = 0$

$+21 \quad +21$

$\frac{3x^2}{3} = \frac{21}{3}$

$\sqrt{x^2} = \sqrt{7}$

$x = \pm\sqrt{7}$

Cube Roots are solutions to the equation $x^3 = b$. Every real number has exactly one real cube root, $\sqrt[3]{b}$.

Break for Practice: Simplify

1. $\sqrt[3]{64} = 4$
2. $\sqrt[3]{0} = 0$
3. $\sqrt[3]{-27} = -3$

Summary: In general we can write the following…

1. An n^{th} root of b is a solution of the equation $x^n = b$.

2. a) If n is even and $b > 0$ (a positive #), there are 2 real n^{th} roots.

   The principal n^{th} root is written $\sqrt[n]{b}$.
   The other n^{th} root is written $-\sqrt[n]{b}$.

   b) If n is even and $b = 0$, there is one n^{th} root: $\sqrt[n]{0} = 0$

   c) If n is even and $b < 0$ (a negative #), there is no real n^{th} root. (no solution)

3. If n is odd, there is exactly one real n^{th} root.

Notation: Each part of the notation has a special name.

Index

$$\sqrt[n]{b}$$

Radical Sign

Radicand

Break for Practice: Simplify

1. $\sqrt{100} = 10$

2. $-\sqrt{100} = -10$

3. $\sqrt[4]{-100}$ → No Solution (even)

4. $\sqrt[4]{0.0081} = 0.3$

5. $\sqrt{(-8)^2} = \sqrt{64} = 8$

6. $\sqrt{-8^2} = \sqrt{-64}$ No Solution

7. $\sqrt[3]{\frac{-27}{64}} = \frac{\sqrt[3]{-27}}{\sqrt[3]{64}} = \frac{-3}{4}$

8. $\frac{2}{-\sqrt{36}} = \frac{2}{-6} = -\frac{1}{3}$

9. $\sqrt[5]{32} = 2$

10. For what values of x does each expression represent a real number?

a) $\sqrt{x+2}$ $x \geq -2$

$x + 2 = 0$
$x = -2$

-3 -2 -1

$\sqrt{-3+2}$ $\sqrt{-1}$ no Solution

$\sqrt{-1+2}$
$\sqrt{1}$
1

b) $\sqrt{3-x}$ $x \leq 3$

$3 - x = 0$
$3 = x$

2 3 4

$\sqrt{3-2}$
$\sqrt{1}$
1

$\sqrt{3-4}$
$\sqrt{-1}$
No Solution

c) $\sqrt[3]{x}$ $\mathbb{R}$

$x = 0$

-1 0 1

$\sqrt[3]{-1}$
-1

$\sqrt{1}$
1

Extended Practice:

Simplify each expression. If the expression does not represent a real number, say so.

1a) $\sqrt{16} = 4$	1b) $-\sqrt{16} = -4$	1c) $\sqrt{-16}$ No Solution	1d) $\sqrt[4]{16} = 2$
2a) $\sqrt{64} = 8$	2b) $\sqrt{-64}$ No Solution	2c) $\sqrt[3]{64} = 4$	2d) $\sqrt[3]{-64} = -4$
3a) $\sqrt{0.01} = 0.1$	3b) $\sqrt{-0.01}$ No Solution	3c) $\sqrt[3]{0.001} = 0.1$	3d) $\sqrt[3]{-0.001} = -0.1$
4a) $\sqrt{0.04} = 0.2$	4b) $-\sqrt{0.04} = -0.2$	4c) $\sqrt{0.0004} = 0.02$	4d) $\sqrt{-0.0004}$ No Solution
5a) $\sqrt{7^2} = 7$	5b) $\sqrt[3]{7^3} = 7$	5c) $\sqrt[4]{(-7)^4} = 7$	5d) $\sqrt[5]{(-7)^5} = -7$
6a) $\sqrt{\frac{1}{64}} = \frac{1}{8}$	6b) $\frac{1}{\sqrt{64}} = \frac{1}{8}$	6c) $\sqrt[3]{-\frac{1}{64}} = -\frac{1}{4}$	6d) $-\frac{1}{\sqrt[3]{64}} = -\frac{1}{4}$
7a) $\sqrt{\frac{1}{16}} = \frac{1}{4}$	7b) $\sqrt{\frac{81}{16}} = \frac{9}{4}$	7c) $\sqrt[4]{\frac{1}{16}} = \frac{1}{2}$	7d) $\sqrt[4]{\frac{81}{16}} = \frac{3}{2}$

Find the **real** roots of each equation. If there are none, say so.

8. $x^2 = 144$ $x = \pm 12$	9. $y^2 - 7 = 0$ $y = \pm\sqrt{7}$
10. $16y^2 = 25$ $y = \pm \frac{5}{4}$	11. $0 = 4 + 16x^2$ No Solution

For what values of x does each expression represent a real number?

12. $\sqrt{x+1}$	13. $\sqrt{x-1}$	14. $\sqrt[3]{x-1}$
$x \geq -1$	$x \geq 1$	$\mathbb{R}$

Simplifying Radicals

One of the first skills that is needed with radicals is knowing how to simplify radicals.

Rules: An expression containing nth roots is in simplest radical form if:

a) No radicand contains a factor (other than 1) that is a perfect nth power.

b) Every denominator has been **rationalized**, so that no radicand is a fraction and **no radical is in the denominator.**

We will begin with the first rule, and we will start with square roots since you may have had some experience with those in the past.

Break for Practice: Simplify

1. $\sqrt{125} \rightarrow \sqrt{25} \cdot \sqrt{5}$

$5\sqrt{5}$

2. $\sqrt{12} \rightarrow \sqrt{4} \cdot \sqrt{3}$

$2\sqrt{3}$

3. $\sqrt{72} \rightarrow \sqrt{36} \cdot \sqrt{2}$

$6\sqrt{2}$

4. $\sqrt{48} \rightarrow \sqrt{16} \cdot \sqrt{3}$

$4\sqrt{3}$

5. $\sqrt{162} \rightarrow \sqrt{81} \cdot \sqrt{2}$

$9\sqrt{2}$

4. $\sqrt{784}$ $= 2 \cdot 2 \cdot 7$

2 392

2 196

2 98

2 49

7 7

$= 28$

Know:

$2^2 = 4$

$3^2 = 9$

$4^2 = 16$

$5^2 = 25$

$6^2 = 36$

$7^2 = 49$

$8^2 = 64$

$9^2 = 81$

$10^2 = 100$

$11^2 = 121$

$12^2 = 144$

Other radicals are more difficult since we are not as familiar with other powers than 2. For this reason, we will use a factor tree method.

Break for Practice: Simplify

1. $\sqrt[5]{64} = 2\sqrt[5]{2}$

2. $\sqrt[3]{1250} = 5\sqrt[3]{2\cdot 5} = 5\sqrt[3]{10}$

3. $\sqrt[3]{640} = 2\cdot 2\cdot \sqrt[3]{2\cdot 5} = 4\sqrt[3]{10}$

4. $\sqrt[3]{3125} = 5\sqrt[3]{5\cdot 5} = 5\sqrt[3]{25}$

5. $\sqrt[3]{5^4 \cdot 2^5 a^4 b^3} = 5\cdot 2ab\sqrt[3]{5\cdot 2^2 a} = 10ab\sqrt[3]{20a}$

6. $\sqrt[5]{3^{12}x^2y^6z^5} = 3\cdot 3yz\sqrt[5]{3^2x^2y} = 9yz\sqrt[5]{9x^2y}$

Extended Practice: Simplify

1. $\sqrt{52} = 2\sqrt{13}$	2. $\sqrt{75} = 5\sqrt{3}$
3. $\sqrt{108} = 6\sqrt{3}$	4. $\sqrt{500} = 10\sqrt{5}$

5. $\sqrt[3]{80}$ = $2\sqrt[3]{10}$	6. $\sqrt[3]{486}$ = $3\sqrt[3]{18}$
7. $\sqrt[4]{486}$ = $3\sqrt[4]{6}$	8. $\sqrt[3]{96}$ = $2\sqrt[3]{12}$
9. $\sqrt[3]{216}$ = 6	10. $\sqrt[4]{3125}$ = $5\sqrt[4]{5}$
11. $\sqrt[3]{2^7a^5b}$ = $4a\sqrt[3]{2a^2b}$	12. $\sqrt[5]{3^{15}xy^6}$ = $27y\sqrt[5]{xy}$
13. $\sqrt{2^5x^3y^2z}$ = $4xy\sqrt{2xz}$	14. $\sqrt[3]{2^5 \cdot 3^4ab^6}$ = $6b^2\sqrt[3]{12a}$

Now before addressing the second rule of simplifying radicals, it is necessary to understand how to multiply radicals. The radicals should have the same index, and then the radicands are simply multiplied together.

Property: $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$

Break for Practice: Simplify

1. $\sqrt{2} \cdot \sqrt{6} = \sqrt{12} \rightarrow \sqrt{4} \cdot \sqrt{3} = 2\sqrt{3}$

2. $\sqrt{10} \cdot \sqrt{15} = \sqrt{150} \rightarrow \sqrt{25} \cdot \sqrt{6} = 5\sqrt{6}$

3. $\sqrt[3]{15} \cdot \sqrt[3]{18} = \sqrt[3]{270} = 3\sqrt[3]{10}$

4. $\sqrt[3]{6} \cdot \sqrt[3]{20} = \sqrt[3]{120} = 2\sqrt[3]{10}$

Mult. CAN distribute

$(5+\sqrt{6})^2$ CANNOT distribute

5. $(5\sqrt{6})^2 = 5^2\sqrt{6}^2 = 25(6) = 150$

6. $(4\sqrt{3})^2 = 4^2\sqrt{3}^2 = 16(3) = 48$

7. $(2\sqrt{3}) \cdot (5\sqrt{5}) = (2)(5) \cdot (\sqrt{3})(\sqrt{5}) = 10\sqrt{15}$

8. $2\sqrt{10} \cdot \sqrt{2} = 2\sqrt{20} \rightarrow \sqrt{4} \cdot \sqrt{5} = 2(2)\sqrt{5} = 4\sqrt{5}$

At this point we are ready to deal with the second rule for simplifying radicals. Remember that this rule states that every denominator has been rationalized, so that no radicand is a fraction and no radical is in the denominator. Once again it is easiest to deal with square roots first, and then we will extend the technique to other radicals.

Break for Practice: Simplify

1. $\frac{2}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{2\sqrt{5}}{\sqrt{25}} = \frac{2\sqrt{5}}{5}$

2. $\frac{15}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{15\sqrt{3}}{\sqrt{9}} = \frac{15\sqrt{3}}{3} = 5\sqrt{3}$

3. $\frac{3}{\sqrt[3]{4}} \cdot \frac{\sqrt[3]{2}}{\sqrt[3]{2}} = \frac{3\sqrt[3]{2}}{\sqrt[3]{8}} = \frac{3\sqrt[3]{2}}{2}$

4. $\frac{1}{\sqrt[5]{8}} \cdot \frac{\sqrt[5]{4}}{\sqrt[5]{2 \cdot 2}} = \frac{\sqrt[5]{4}}{\sqrt[5]{32}} = \frac{\sqrt[5]{4}}{2}$

5. $\frac{2}{\sqrt[4]{125}} \cdot \frac{\sqrt[4]{5}}{\sqrt[4]{5}} = \frac{2\sqrt[4]{5}}{\sqrt[4]{625}} = \frac{2\sqrt[4]{5}}{5}$

6. $\frac{\sqrt{4}}{4\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{2\sqrt{5}}{4\sqrt{25}} = \frac{\sqrt{5}}{2(5)} = \frac{\sqrt{5}}{10}$

Extended Practice: Simplify

1. $\sqrt{3} \cdot \sqrt{15}$ $= 3\sqrt{5}$	2. $\sqrt{6} \cdot \sqrt{8}$ $= 4\sqrt{3}$
3. $\sqrt[3]{25} \cdot \sqrt[3]{10}$ $= 5\sqrt[3]{2}$	4. $\sqrt[3]{8} \cdot \sqrt[3]{8}$ $= 4$
5. $(7\sqrt{2})^2$ $= 98$	6. $(5\sqrt{3})^2$ $= 75$
7. $(2\sqrt{2})(5\sqrt{8})$ $= 40$	8. $\frac{7}{\sqrt{2}}$ $= \frac{7\sqrt{2}}{2}$
9. $\frac{6}{\sqrt{3}}$ $= 2\sqrt{3}$	10. $\frac{15}{\sqrt{5}}$ $= 3\sqrt{5}$

11. $\frac{3}{\sqrt[4]{8}}$ $= \frac{3\sqrt[4]{2}}{2}$	12. $\frac{8}{\sqrt[3]{25}}$ $= \frac{8\sqrt[3]{5}}{5}$
13. $\frac{6}{\sqrt[5]{8}}$ $= 3\sqrt[5]{4}$	14. $\frac{12}{\sqrt[4]{9}}$ $= 4\sqrt[4]{9}$
15. $\frac{9\sqrt{2}}{\sqrt{18}}$ $= 3$	16. $\frac{\sqrt{4}}{2\sqrt{20}}$ $= \frac{\sqrt{5}}{10}$

Sums and Differences of Radicals

In this section we will see how to deal with expressions involving the sums and differences of radicals. In many ways, it is similar to combining like terms with algebraic expressions**. Radicals can be added or subtracted when their indices and radicands are the exact same.**

Review: Like Terms → $2x^2y$, $3x^2y$

exact same variable/exponent combo

Like Radical: $\sqrt[3]{2}$, $-2\sqrt[3]{2}$

(Same Index)

Same Radicand

$6\sqrt[4]{5}$, $15\sqrt[4]{5}$

Break for Practice: Simplify 1st, then add/subtract

1. $\sqrt{27} + \sqrt{12}$ $= 3\sqrt{3} + 2\sqrt{3}$

$\sqrt{9}\sqrt{3}$ $\sqrt{4}\sqrt{3}$

$= 5\sqrt{3}$

2. $\sqrt{28} + \sqrt{63}$ $= 2\sqrt{7} + 3\sqrt{7}$

$\sqrt{4}\sqrt{7}$ $\sqrt{9}\sqrt{7}$

$= 5\sqrt{7}$

3. $\sqrt{32} - \sqrt{50} + \sqrt{98} = 4\sqrt{2} - 5\sqrt{2} + 7\sqrt{2}$

$\sqrt{16}\sqrt{2}$ $\sqrt{25}\sqrt{2}$ $\sqrt{49}\sqrt{2}$

$= 6\sqrt{2}$

4. $\sqrt[3]{54} - \sqrt[3]{16} + \sqrt[3]{27} = 3\sqrt[3]{2} - 2\sqrt[3]{2} + 3$

$\sqrt[3]{27}\sqrt[3]{2}$ $\sqrt[3]{8}\sqrt[3]{2}$

$= \sqrt[3]{2} + 3$

5. $\sqrt[3]{135} - \sqrt[3]{40} + \sqrt[3]{2} = 3\sqrt[3]{5} - 2\sqrt[3]{5} + \sqrt[3]{2}$

5 27   2 20

3 9   2 10

3 3   2 5

$= \sqrt[3]{5} + \sqrt[3]{2}$

6. $\sqrt{2}(\sqrt{32} + \sqrt{12}) = \sqrt{64} + \sqrt{24} \to \sqrt{4}$ $\to \sqrt{6}$

$= 8 + 2\sqrt{6}$

7. $2\sqrt{3}(4\sqrt{2} - 5\sqrt{12}) = 8\sqrt{6} - 10\sqrt{36}$

$= 8\sqrt{6} - 10(6)$

$= 8\sqrt{6} - 60$

Extended Practice: Simplify

1. $\sqrt{50} + \sqrt{18} = 8\sqrt{2}$	2. $3\sqrt{12} - \sqrt{48} = 2\sqrt{3}$
3. $\sqrt{27} + 2\sqrt{75} = 13\sqrt{3}$	4. $5\sqrt{2} - 2\sqrt{5}$ Cannot be simplified further

5. $\sqrt{50}+\sqrt{63}-\sqrt{32}$ $= 3\sqrt{7}+\sqrt{2}$	6. $\sqrt{18}+\sqrt{24}-\sqrt{54}$ $= 3\sqrt{2}-\sqrt{6}$
7. $\sqrt[3]{54}+\sqrt[3]{40}+\sqrt[3]{16}$ $= 2\sqrt[3]{5}+5\sqrt[3]{2}$	8. $\sqrt{2}(\sqrt{8}+\sqrt{10})$ $= 4+2\sqrt{5}$
9. $\sqrt{3}(\sqrt{12}-\sqrt{24})$ $= 6-6\sqrt{2}$	10. $3\sqrt{5}(\sqrt{5}+2\sqrt{75})$ $= 15+30\sqrt{15}$

Binomials Containing Radicals

In this section we will see how we can multiply and divide binomials that contain radicals. The process is very similar to what we used with algebraic binomials in the past.

How do you think you would multiply something like this?

$(2+\sqrt{5})(3-\sqrt{5}) = \underset{F}{6} \underset{O}{-2\sqrt{5}} \underset{I}{+3\sqrt{5}} \underset{L}{-\sqrt{25}}$

$= 6+\sqrt{5}-5$

$= 1+\sqrt{5}$

Break for Practice: Simplify

1. $(3+\sqrt{2})(\sqrt{3}-4)$ $= 3\sqrt{3}-12+\sqrt{6}-4\sqrt{2}$

2. $(2\sqrt{5}+3)(\sqrt{5}-4)$ $= 2\sqrt{25}-8\sqrt{5}+3\sqrt{5}-12$

$= 2(5)-5\sqrt{5}-12$

$= 10-5\sqrt{5}-12$

$= -5\sqrt{5}-2$